OBSERVER-BASED PARAMETER ESTIMATION AND FAULT DETECTION

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# Contents

Abstract 9

Declaration 11

Copyright 12

Acknowledgements 13

Notations 14

Publications 16

## 1 Introduction

1.1 Overview of model based fault detection .......................... 17
1.2 Motivation for the study ............................................. 19
1.3 Thesis outline ......................................................... 21
1.4 Thesis contribution .................................................. 22

## 2 Literature review

2.1 Dynamic modelling of aero engines ................................. 24
2.2 Parameter estimation of Output Error model ....................... 26
2.3 Model-based fault detection ........................................... 27
  2.3.1 Parameter estimation based fault detection .................... 30
  2.3.2 Observer-based fault detection .................................. 30
2.4 Dynamic observer design ............................................. 33

## 3 Nonlinear Least Squares for OE Model Identification

3.1 Problem formulation .................................................. 36
  3.1.1 Criterion selection ................................................ 37
# Model selection: EE v.s. OE

- 3.1.2 Model selection: EE v.s. OE

## NLS for OE parameter identification

- 3.2 NLS for OE parameter identification
  - 3.2.1 Calculation of $\partial V(\theta_k)/\partial \theta$ and the Jacobian
  - 3.2.2 Approximation of $R(k)$ and the Hessian

## Application and results

- 3.3 Application and results
  - 3.3.1 First-order model
  - 3.3.2 Second-order model

## Summary

- 3.4 Summary

---

# High-gain Observer-based Parameter Estimation

- 4 High-gain Observer-based Parameter Estimation
  - 4.1 Introduction
  - 4.2 High-gain observer for disturbance estimation
  - 4.3 High-gain observer based parameter estimation
  - 4.4 Finite State Machine (FSM)-based adaptive change detection
    - 4.4.1 Adaptive change detection
    - 4.4.2 Finite State Machine implementation
  - 4.5 Case study: a gas turbine engine system
  - 4.6 Summary

---

# Parameter Estimation with Delay Compensation

- 5 Parameter Estimation with Delay Compensation
  - 5.1 Introduction
  - 5.2 TFMs in disturbance estimation
    - 5.2.1 TFM $G_s$ (s) relating $u(t)$ to $\dot{d}(t)$
    - 5.2.2 TFM $H_{du}(s)$ relating $u(t)$ to $d(t)$
  - 5.3 Independence of time delay
  - 5.4 Delay compensation by filter approximation
    - 5.4.1 Phase delay
    - 5.4.2 Magnitude-phase approximation
    - 5.4.3 Parameter estimation with delay compensation
  - 5.5 Case Study: a servo motor system
  - 5.6 Summary

---

# Robust Static Fault Detection Observer Design

- 6 Robust Static Fault Detection Observer Design
  - 6.1 Introduction
    - 6.1.1 On-board condition monitoring of GTEs
    - 6.1.2 Robust fault detection observers
List of Tables

3.1 Comparison on training data set (1st order model) ............... 50
3.2 Comparison on validation data set (1st order model) ........... 53
3.3 Comparison for 2nd order model ............................. 54

4.1 Comparison of different methods (noise free) .................. 73
4.2 Comparison of different methods (subject to noises) ........... 75

5.1 Delay comparison of different $[\Delta A, \Delta B] (\mu = 25)$ .......... 101
5.2 Comparison of different methods .................................. 104

A.1 Gas turbine engine nomenclature ............................... 175
## List of Figures

2.1 Dual-lane control of aero engines. ........................................ 25  
2.2 Classification of fault detection approaches .......................... 28  
3.1 Comparison of Long-step prediction and one-step-ahead prediction 37  
3.2 Residuals of long-term prediction and one-step ahead prediction  . 38  
3.3 Definitions of Output Error and Equation Error. ....................... 39  
3.4 Scatter plot of the output $y(t)$ versus the input $u(t-1)$. ....... 48  
3.5 The objective function of OE model ................................. 49  
3.6 Search route of BFGS ............................................. 51  
3.7 Search route of DNLS ............................................ 51  
3.8 Long-term predictions of DNLS (first-order model) ............... 52  
3.9 Long-term predictions of DNLS (second-order model) ............ 53  
4.1 A FSM presentation: flow chart .................................... 66  
4.2 Sateflow® based adaptive change detection .......................... 68  
4.3 The input signal to the gas turbine engine and its spectrum ... 70  
4.4 State estimation of the gas turbine engine (noise free). ........... 71  
4.5 The actual disturbance $d(t)$ and its estimate (noise free) ....... 71  
4.6 The actual parameter variation $[\Delta A, \Delta B]$ and its estimate $[\hat{\Delta A}, \hat{\Delta B}]$ (noise free). ................................. 72  
4.7 States estimation of the gas turbine engine (corrupted by noises). 73  
4.8 The actual disturbance $d(t)$ and its estimate (corrupted by noises) 74  
4.9 The actual parameter variation $[\Delta A, \Delta B]$ and its estimate (corrupted by noises) ................................. 74  
5.1 Decomposition of the plant and its high-gain observer. ............ 82  
5.2 The scheme of delay compensation and parameter estimation ... 88  
5.3 The input $u(t)$ and its spectrum. ................................ 97  
5.4 Magnitude response of $F(s)$ ........................................ 99
### 5.5 Phase response of $F(s)$

5.6 Time delay $\tau_i(\omega)$ relating $d_i(t)$ to $\hat{d}_i(t)$, $i = 1, 2$.  

5.7 Disturbance $d(t)$ and its estimate $\hat{d}(t)$ ($\mu = 25$).  

5.8 Bode plots of $F_{ii}(s)$ (blue) and its compensation filter $L_i(s)$ (red).  

5.9 The phase delay of $F_{ii}(s)$ and $L_i(s)$, and the approximation errors.  

5.10 Disturbance estimation $\hat{d}_1(t)$, and the compensated $u(t)$, $\hat{x}_1(t)$ and $\hat{x}_2(t)$ by $L_1(s)$ for identifying $\theta_1$.  

5.11 Disturbance estimation $\hat{d}_2(t)$, and the compensated $u(t)$, $\hat{x}_1(t)$ and $\hat{x}_2(t)$ by $L_2(s)$ for identifying $\theta_2$.  

5.12 The estimated parameter variations by Algorithm 5.3.  

### 6.1 Observer based on-board condition monitoring of GTEs.  

6.2 Disturbances injected into the system.  

6.3 The input, outputs and output estimates predicted by the observer $K_0$.  

6.4 The actual disturbance spectrum and its estimation through 128-point FFT-based residuals spectrum analysis.  

6.5 Residuals corresponding to $K_{opt}$, $K_{place}$ and $K_{inf}$ respectively (left column) and their spectra (right column) (fault free).  

6.6 Residuals of $K_{opt}$, $K_{place}$ and $K_{inf}$ in the case of a saturated incipient actuator fault.  

### 7.1 Structure of the observer-based fault detection.  

7.2 The largest singular values and smallest singular values of $\tilde{G}_d(s)$ and $\tilde{G}_i(s)$, respectively.  

7.3 Quasi-stationary disturbances.  

7.4 Spectra of the disturbances and the residuals of $K_0$.  

7.5 Comparison of magnitude responses of TFMs $\tilde{G}_f(s)$ and $\tilde{G}_d(s)$.  

7.6 Fault free residuals corresponding to the dynamic observer, the $H_\infty/H_\infty$ observer $L_1$ and static observer $L_2$, respectively.  

7.7 Residuals of the dynamic observer, $H_\infty/H_\infty$ observer $L_1$ and static observer $L_2$, respectively, for detecting abrupt actuator faults.  

7.8 Plant outputs and residuals under an incipient actuator fault at the first input channel.  

### A.1 Schematic diagram of the Spey gas turbine engine.
Abstract

This PhD work is motivated by on-board condition monitoring of gas turbine engines (GTEs) and presents a constructive robust fault detection procedure integrating system identification, time delay compensation, eigenstructure assignment, zero assignment and dynamic observer design techniques, to detect faults in a dynamic system corrupted by disturbances at some frequencies. The main results achieved in this PhD study are:

1. Application of nonlinear least squares to Output Error (OE) model identification. Although OE model shows better performance on long-term prediction, the challenge is the dependency within the long-term prediction errors. The dependency is tackled by an iterative calculation of the gradient, and an approximation of the Hessian matrix is adopted to accelerate the convergence.

2. Delay compensation for high-gain observer based time-varying parameter estimation. In the high-gain observer based parameter estimation, it is usually assumed that the estimation delay is zero. This assumption puts some constraints on the observer design and may not be satisfied in some situations. By examining the transfer function matrices associated with the high-gain observer, a novel time delay calculation and compensation approach is proposed. The main contribution is the proof of the fact that the estimation delay is free from the plant parameter variation. Then a nonlinear phase delay filter approximation technique is used to compensate the delay.

3. Zero assignment in dynamic fault detection observer design. It is well known in filter design that zeros have the ability to block the propagation of some input signal through the system at some frequency. In this thesis, this idea is used to assign zeros to the desired places so that the disturbance can be attenuated. In most observer design research, however, the structure is confined to the classic (static) Luenberger structure where the gain is a constant numerical matrix. As proved in this thesis, zeros of static observers are invariant. Hence, the
dynamic observer is proposed, where a dynamic system (dynamic feedback gain) substitutes for the constant numerical gain matrix. As a result, some additional zeros are introduced and can be assigned arbitrarily to the desired places. To the best of our knowledge, although the concept of zeros in multivariable systems has been proposed by Rosenbrock over thirty years, there have been no known results of utilising zero assignment to robust fault detection observer design.
Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.
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Further information on the conditions under which disclosures and exploitation may take place is available from the Head of School of Electrical and Electronic Engineering.
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Notations

Abbreviations

ARX      AutoRegressive eXogeneous (model)
EE       Equation Error
FDI      Fault Detection and Isolation
FFT      Fast Fourier Transform
FSM      Finite State Machine
GTE      Gas Turbine Engine
IIR      Infinite Impulse Response (Filter)
LSE      Least Squares Estimation
FDI      Fault Detection and Isolation
NLPD     NonLinear Phase Delay
M-NLPD   Magnitude and Nonlinear Phase Delay
NLS      Nonlinear Lease Squares (Estimation)
DNLS     Dynamic Nonlinear Least Squares (Estimation)
OE       Output Error
RFDO     Robust Fault Detection Observer
TDE      Time Delay Estimation
TFM      Transfer Function Matrix
UIO      Unknown Input Observer

Spaces, sets and points

$\mathbb{R}$  Space of real numbers
$\mathbb{R}^n$  $n$-dimensional real space
$\mathbb{C}^n$  Space of complex numbers
$\mathbb{C}^n$  $n$-dimensional complex space
$[a, b]$  Closed interval of $\mathbb{R}$
$(a, b)$  Open interval of $\mathbb{R}$

Mathematical Operators


max $f(\theta)$ $\quad$ Maximum of the function $f(\theta)$
$max_{\theta \in [a, b]} f(\theta)$ $\quad$ Maximum of the function $f(\theta)$ over $[a, b]$
$\min$ $\quad$ Minimum value of a set
$\sup f(x)$ $\quad$ Supremum of the function $f(x)$
$|f(x)|_{x=a}$ $\quad$ Absolute of function $f(x)$ when $x = a$
$\|A\|_1$ $\quad$ 1-norm of $A$
$A^T$ or $A'$ $\quad$ Transpose of $A$
$a_{ij}$ $\quad$ The $(i, j)^{th}$ element of $A$
$h(t) * n(t)$ $\quad$ convolution product of signal $h(t)$ and $n(t)$
$\|f(x)\|_{x=a}$ $\quad$ the norm of function $f(x)$ when $x = a$

General Symbols, unless stated otherwise
$A, B, C, D$ $\quad$ Matrices of the plant
$A_0, B_0, C_0, D_0$ $\quad$ Matrices of the nominal model for the plant
$\Delta A, \Delta B, \Delta C, \Delta D$ $\quad$ parameter variations
$n$ $\quad$ Dimension of the system
$p$ $\quad$ Dimension of the input
$r$ $\quad$ Dimension of the output
$m$ $\quad$ Dimension of the dynamic observer feedback
$x(t)$ $\quad$ Signal/variable in continuous domain
$x(k)$ $\quad$ Signal/variable in discrete domain
$\theta$ $\quad$ Parameters to be identified
$\text{eig}(A)$ $\quad$ Eigenvalues of matrix $A$
$\lambda$ $\quad$ A pole/eigenvalue of a system
$\Lambda$ $\quad$ Set of eigenvalues of a system
$\omega$ $\quad$ Frequency vector
$Z$ $\quad$ Set of system zeros
$K$ or $L$ $\quad$ Gain matrix of the observer
$G(s)$ $\quad$ Transfer function matrix of the plant
$K(s)$ $\quad$ Transfer function matrix of the observer feedback
$\|G(s)\|$ $\quad$ Norm of Transfer function matrix $G(s)$
$\|G(s)\|_{\infty}$ $\quad$ $H_\infty$-norm of Transfer function matrix $G(s)$
$\|G(s)\|_2$ $\quad$ 2-norm of Transfer function matrix $G(s)$
$\|G(s)\|_{s=j\omega}$ $\quad$ Norm of numerical $G(s)$ evaluated at point $s = j\omega$
Publications

Journal Papers


Conference Papers


Chapter 1

Introduction

1.1 Overview of model based fault detection

Although the increasing demand for higher safety and reliability has stimulated an extensive study of fault detection and isolation (FDI) since the early 1970s, FDI is still very young compared to control engineering. As FDI continues to mature, it has been an important and fruitful part of modern control, and an abundance of results have been reported ranging from physical redundancy, analytical redundancy to statistical learning and artificial intelligence.

In most FDI methods, the basis is information redundancy. A traditional approach is physical redundancy requiring at least a dual sets of redundant physical devices, but it faces the problem of extra hardware costs and additional weight and space (these are particularly of economical interest in aero engines). Analytical redundancy uses a mathematical model to mimic the actual system’s behaviour. From the viewpoint of modelling, analytical model-based FDI methods require either quantitative models or qualitative models. In quantitative modelling, the plant is expressed in terms of mathematical functional relationships between the input and output variables. However, it suffers from the need of an accurate model. In qualitative model, these relationships are expressed in terms of qualitative functions, where a priori knowledge about the model is assumed. On the other hand, if a qualitative model is still difficult to reach, history-based method can be used where only the large amount of historical process data is required. This method requires some ‘intelligent’ techniques, such as data-mining, feature extraction, pattern recognition and classifier design. These
later two methods require higher level of computation, so are not usually suitable for on-board condition monitoring. In this PhD study we concentrate on quantitative model-based parameter estimation and residual generation.

In the field of (quantitative) model-based fault detection, three main approaches have been developed: (1) parameter estimation method resting on system identification [1, 2]; (2) parity relation approach [3, 4, 5, 6]; (3) observer/filter-based approach [7, 8]. At the beginning, the model-based fault detection was considered in control design, where Ratner and Luenberger [2] propose an adaptive control design for systems with parameter jumps abruptly. Later, Beard etc. [10], [11] proposed the so-called failure detection filter for linear systems and a survey paper on the observer(filter) approach in a stochastic framework was given by Willsky [12]. Since 1980s, the survey papers [1, 2, 13, 14, 7, 15, 16, 8] by Isermann, Gertler, Patton, Frank and Ding, respectively, give a good overview of modern model-based fault detection methods. In 1999, a unified framework of model-based FDI was presented in a book [17] by Chen providing a comprehensive foundation of model-based FDI. Recently, Isermann summarised the state of the art in the field of FDI [18].

The heart of model-based FDI is the generation of residuals, and the robustness has become the main theme of observer(filter)-based methods [19]. As one of dominant approaches, the de-coupling approach has been developed during last two decades. The idea of de-coupling is to treat model uncertainties as some exo-terical disturbances (unknown inputs) and de-couple them from residuals. UIOs (Unknown Input Observers), originally proposed in [20] and first employed for FDI in [21] is to make the state estimation error de-coupled from disturbances and the robust residual generation is achieved indirectly from those disturbance-free states. However, in the context of FDI, it is not necessary to decouple the state estimation from disturbances. A direct approach is to de-couple the residual from disturbances straightforward. A common direct approach is the eigenstructure assignment-based UIO proposed in 1990s [22]. The existence conditions for complete decoupling have been derived both in the indirect approach [23], [21] and in the direct eigenstructure assignment approach [24], [25]. However the perfect de-coupling may not be possible, in some cases, due to the lack of design freedom. Moreover, it may be problematic because the fault effect may also be de-coupled. If the required sufficient condition is not met, an approximate de-coupling should be taken, where the residual is not perfectly de-coupled from disturbances, but
has a low sensitivity to disturbances and high sensitivity to faults. As a result, it becomes an optimisation problem and has been studied in time domain \cite{26, 27, 28}, in frequency domain \cite{29, 30}, via multi-objective optimisation \cite{31, 32, 33}. Meanwhile, stimulated by the $H_\infty$ control that firmly roots in the consideration of model uncertainties, the use of $H_\infty$ optimisation \cite{29, 34, 35, 36} and LMI \cite{37, 38, 39, 40, 41} for robust residual generation has received more attention recently.

Unfortunately, the observers (filters) involved in these methods are confined to the traditional Luenberger observer (Kalman Filter) structure with a fixed (adjustable) numerical feedback gain matrix. From the viewpoint of frequency domain, such a feedback gain does not have the ability of frequency shaping and puts some limitation on disturbance attenuation. It is of interest to change the observer structure and see whether this would be used to improve the fault detection performance further. This is very much the aim of this research, and, here it is mainly the technique of delay compensation in high-gain observer and zero assignment in dynamic observer to be investigated.

### 1.2 Motivation for the study

This PhD study is stimulated from model-based on-board condition monitoring and fault detection of aero gas turbine engines (GTEs). Due to the degradation, a model reflecting the current operating point need to be first built by system identification techniques, then some residual generator is designed for giving fault indication signals. The primary objective of this study can be stated as: build a constructive robust fault detection procedure integrating system identification, eigenstructure assignment, zero assignment and dynamic observer design techniques.

In the modelling stage, the aim is to identify an Output Error (OE) model from the real aero engine input-output data for a better long-term prediction. The real engine parameters may deviate from the general engine model due to individual engine characteristics and degradation. Hence, on-board identification is required during the normal engine operation. Although OE model shows a better performance than Equation Error (EE) model, it challenges to the dependency within the prediction errors making the prediction error non-Gaussian. Furthermore, such an identification algorithm should be fast in terms of on-board
computing. These challenges stimulated the proposition of an iterative calculation of the gradient as shown in Chapter 3.

In the model-based fault detection stage, there have been three main approaches (parameter estimation, observer design, parity space), and two of them are examined here:

1. **high-gain observer based parameter estimation**

   As a model-referenced parameter estimation technique, the high-gain observer based-parameter estimation was recently proposed by Gao et al. [42, 43]. When applied to engine parameter estimation, it was found that the gain have to be selected large enough to avoid incorrect estimates. A further study showed that this is caused by the time delay in disturbance estimation. This discovery conflicts with the general assumption that the high-gain observer estimates the state/disturbance without time-delay. Therefore, in order to relax the constraints on the gain selection and improve the estimation performance, the properties of such a high-gain observer need to be examined to find a solution to either remove or compensate the delay. Due to the existence of exoteric disturbances/noises, the widely used time-delay estimation (TDE) failed. Hence, a new solution is required and will be developed in Chapter 5.

2. **zeros in dynamic observer-based residual generation**

   As presented in the brief review of observer-based fault detection, the involved observers (filters) are restrained in the traditional Luenberger/Kalman structure with a fixed/adjustable numerical feedback gain matrix. From the viewpoint of frequency domain, such a feedback gain does not have the ability of frequency shaping. It is of interest to change the observer structure and see how this would improve the fault detection performance. Inspired by the idea of using zeros to improve signal-noise ratio in the field of communication filter design (mainly on SISO systems), it is felt that zeros in multivariable system may have the same ability. Unfortunately, it is found that zeros in a static observer are invariant, and one cannot change the zeros. Then, a more general framework of observer - dynamic observer - is proposed in Chapter 7.
1.3 Thesis outline

During the course of this PhD study, a number of related topics have been investigated, from introduction of gas turbines to system identification, from delay compensation to dynamic observer-based fault detection. Initial interests focused on real-data driven reduced-order modelling of gas turbine engines and using long-term prediction residual to detect sensor faults. Later, research were concentrated on observer-based time-varying parameter estimation and dynamic observer design.

Each chapter is described in detail at the beginning of the chapter. Here, a general overview of this thesis is presented.

Chapter 2 Literature Review gives a review of the field of observer-based parameter estimation and residual generation for fault detection, starting by Section 2.1 on dynamic modelling of aero engines. Section 2.2 summarises individual studies and articles on OE model identification. Section 2.3 addresses basic techniques for high-gain observer based parameter estimation and observer based fault detection. Section 2.4 gives the background of dynamic observer design and zeros in multivariable systems.

Chapter 3 Nonlinear Least Squares for OE Model Identification presents the modelling phase of this PhD work. A nonlinear least squares method is applied to identifying the reduced order OE model of a aero engine. In this chapter, the identification speed is accelerated by (1) an iterative calculation of the gradient, and, (2) the Hessian approximation. The dependency within output errors is tackled by the iterative calculation of gradient, which is the main contribution of this chapter.

Chapter 4 High-gain Observer-based Parameter Estimation describes the application of high-gain observer to time-varying parameter estimation. With such a high-gain observer, the states and model disturbances are estimated simultaneously and the parameter variation can be estimated within an ARX framework. Furthermore, Section 4.4 presents a Finite State Machine (FSM)-based adaptive change detection approach to detect the change of parameter values. This technique is designed to improve the parameter estimate. The simulation results on a aero engine in Section 4.5 illustrates the application of this technique to time-varying parameter estimation. Note that in this chapter it is assumed that the disturbance estimation delay, (the time delay between the disturbance and its estimate) is equal to zero.
Chapter 5 Parameter Estimation with Delay Compensation gives a more general treatment of nonzero disturbance estimation delay in high-gain observer based parameter estimation. The main contribution is the constructive proof of the independency of delay from the parameter variation. Hence, in Section 5.3 a novel analytical delay calculation algorithm is proposed, and in Section 5.4 a nonlinear phase delay filter approximation technique is used to compensate the delay. As illustrated in the simulation results on a servo motor, such techniques not only improve the performance of the high-gain observer based parameter estimation, but also give a new insight into the high-gain observer design.

Chapter 6 Robust Static Fault Detection Observer Design focuses on discrete robust fault detection design for on-board condition monitoring of gas turbine engines. Because of the limited on-board computation resources, a fast design method is required. This objective is achieved by (1) a Fast Fourier Transform based disturbance frequency estimation; (2) a modified robustness/sensitivity index by incorporating such frequency information; (3) an optimal eigenvalue assignment method for observe gain selection.

Chapter 7 Dynamic Observer Design for Fault Detection proposes a new observer structure (dynamic observer) for fault detection. First of all, the so-called dynamic observer differs from the traditional Luenberger observer (Kalman filter) by using a dynamic feedback gain system, rather than a numerical matrix. This chapter proves that the dynamic feedback gain introduces additional zeros to the observer. Then, a novel zero assignment approach is proposed to attenuate disturbances by assigning observer zeros to the disturbance frequencies. Note that the zero assignment is only available in dynamic observers. The zeros assignment for dynamic observer design is the main contribution of this thesis.

The final part is an appendix of modelling of aero gas turbine engines.

1.4 Thesis contribution

The main contributions of this PhD study can be stated as follows:

1. delay compensation for high-gain observer based time-varying parameter estimation
In order to improve the parameter estimation, the Transfer Function Matrices (TFMs) associated with disturbance estimation in the high-gain observer are examined, and a novel time delay calculation and compensation approach is proposed. The main result is that the estimation delay is free from the parameter variation (see Theorem 5.2 and its propositions in Chapter 5). Then the delay is compensated by using a nonlinear phase delay approximation filter to lag the state estimates by the same delay, because it is impossible to lead a non-periodical signal in a causal system. The advantage of this delay computation and compensation algorithm is its insensitivity to external disturbances and measurement noises.

2. zero assignment for dynamic observer-based fault detection

There are two main contributions in the proposed dynamic observer design: (1) Stimulated by the dynamic feedback control, an extended observer structure is proposed for observer-based fault detection. In most observer design, the observer is confined to the traditional (static) Luenberger structure where the gain is a constant numerical matrix without the ability of frequency shaping. In Chapter 7 a dynamic observer is proposed by replacing the constant gain matrix with a dynamic system. Therefore, a better fault detection performance can be achieved by shaping the frequency response of the dynamic feedback gain. (2) The introduction of dynamic gain matrix also brings additional zeros to the dynamic observer. Note that, these additional zeros can be assigned arbitrarily. Thus, the zero invariance in static observers is overcome and one can put zeros close to the disturbance frequencies to attenuate the disturbances. To the best of our knowledge, there have been no known results of utilising zero assignment to design an observer, although the concept of zeros in multivariable systems has been proposed and the signal blocking properties of zeros have been studied by Rosenbrock etc. over thirty years.
Chapter 2

Literature review

This chapter gives a brief introduction to the field of parameter identification and model-based fault detection, starting by Section 2.1 on dynamic modelling of aero engines. Section 2.2 summarises individual studies and articles on OE model identification. Section 2.3 addresses basic techniques for high-gain observer based parameter estimation and robust fault detection observer design. Section 2.4 gives the background of dynamic observer design and zeros in multivariable systems. The purpose is merely to introduce concepts and ideas from which subsequent chapters can depart.

2.1 Dynamic modelling of aero engines

As shown in the Appendix A due to various aims of GTE modelling, a lot of methods have been proposed and employed to set up different models, e.g. non-linear thermodynamic models for engine design, nonlinear static/dynamic models for demonstration/tuning, linear dynamic models for controller design, real-time piecewise linear models for on-board application.

From the viewpoint of condition monitoring, control systems in modern aero engines are usually organised as dual-lane systems with two sets of parallel sensors and controllers [44]. One lane works as primary lane to issue control signals, another is waiting in "hot back-up", as illustrated in Figure 2.1. An information redundancy exists by using such two sets of duplicated hardware, which can be used for fault detection.

However, this hardware redundancy cannot decide which lane is in failure. It is necessary to introduce a third lane and perform a majority vote scheme.
Unfortunately, having three or more hardware lanes would be very expensive and could increase aircraft weight significantly. Furthermore, the hardware redundancy does not provide sufficient information to detect faults in the engine itself. A long-term prediction model, some sort of mathematic redundancy, can be used as the third lane to solve this problem [44].

**Why reduced order model**

The total duration of fault detection and lane switching must not exceed the critical time limit. This requires that the model should be simple enough for computing in real-time. Although full thermodynamic nonlinear models are the most accurate models, they tend to be too complicated to compute in real-time. Because the aero engine operates at a number of operating points and a linear model usually embraces 3-5% of controlled coordinates [45], a practical way is to combine the linear dynamic model with the nonlinear static model. In recent years, the increase of computation power makes it possible to use system identification techniques for fitting parametric models to real data. Therefore, the reduced order dynamic modelling for fault detection has gained more interests from both academic research and industrial applications.

Theory study on reduced order modelling can be found in [46], [47], [48], [49], [50]. Many data-driven approaches have been applied in on-board model identification and condition monitoring [51], [52], [53]. The papers [54], [55] examined the time domain methods for estimating discrete models. Some researchers made use of Genetic Algorithms to optimise the performance of model prediction [44], [45]. Recent work by [56], [57], [58], [53] studied the application of frequency-domain identification techniques to dynamic modelling of GTEs.
Spectral estimation methods are used to improve the identification accuracy of FRF (Frequency-Response Function) in [45], [59], [60]. In stochastic methods, Markov models [61], [62] and Genetic Algorithms [53], [62], neural networks [63] are studied. In [45], [60], RPLDM (Real-time Piecewise Linear Dynamic Model) and LDM (Linear Dynamic Model) are applied for real-time modelling. In [27], [47], reduced order models are employed for condition monitoring.

2.2 Parameter estimation of Output Error model

Basically, there are two connection modes used for condition monitoring [64]:

1. parallel connection for long-term prediction (which corresponds to the "Output Error (OE)" model or "Infinite Impulse Response (IIR)" filter);

2. parallel-series connection for one-step-ahead prediction (which corresponds to the "Equation Error" model or "Finite Impulse Response (FIR)" filter).

It has been recognised that, in the context of condition monitoring, the long-term prediction performance is more of interest than the one-step-ahead prediction [44], [65], [54], [55]. Unfortunately, it is well known that the LSE method may lead to biased parameter estimates, due to the correlated residual (output error) [66], [67], [68], [69], [70]. On the other hand, although OE model is unbiased, its objective function is a highly nonlinear function, rather than a quadratic function as in LSE [67], [26], [51].

The OE model identification was first treated systematically by Landau using a model reference adaptive technique [71] and two recursive algorithms were proposed in [66]. In parallel, the problem of A-IIR (Adaptive Infinite Impulse Response) filtering was studied in the field of adaptive signal processing [72]. Feintuch originally proposed the LMS (Least Mean Square) algorithm [73], and Johnson, Larimore etc. proposed an adaptive recursive filter SHARF [74]. In fact, A-IIR filtering is equivalent to OE modelling, and those researchers just studied the same problem from different viewpoints. From a personal viewpoint, it is felt that A-IIR filtering is an application of OE modelling in signal processing. One interesting thing is that, in recent years, more activities were carried out in the field of adaptive signal processing. One can see many papers were published in terms of ‘adaptive filtering’, rather than OE identification [72], [75], [67], [66], [71], [78], [79]. It seems that control people have lost interests on this problem,
possibly because the OE modelling is more application oriented and too simple in theory.

The simplest way to identify the parameters of a linear system is LSE, but it has been shown that the LSE is biased \[71\], \[67\] giving a poor long-term prediction. Due to the nonlinearity of the mean-square output error (MSOE) objective function, some iterative search algorithm has to be used. Many researchers have suggested to use gradient search method to find the minimum directly, which is the so-called (direct) OE algorithm. Thus the convergence properties of the output error algorithm are dependent on the MSOE surface. It is worth to note that the MSOE surface may has multiple local minima that affects the convergence of the gradient search, and in most cases, the global minimum solution is unbiased even in presence of disturbance \[80\] \[81\]. In practice, the small step approximation is adopted. Another modification is the so-called adaptive LMS in \[73\], \[82\] or pseudo-linear regression in \[68\]. Landau developed algorithms for off-line OE identification based on the hyperstability theory \[83\] \[84\] \[85\] \[66\] and Johnson etc. proposed SHARF (a short for simple hyperstable algorithm for recursive filters) \[74\], where a moving average filter is adopted to characterise the non-Gaussian output error. Because the original SHARF algorithm requires strictly positive realness condition on the unknown moving average filter, some modified SHARF algorithms have been proposed by using a time-varying moving average filter \[86\]. In parallel, many researchers tried to modify the LSE algorithm by removing its bias. Steiglitz and McBride proposed the so-called SM algorithm to overcome the drawback of biased LSE algorithm \[87\]. More examples are bias remedy least-mean-square equation error (BRLE) algorithm developed in \[88\] \[70\], Composite Regressor algorithm \[89\], Composite Error \[90\] \[97\].

2.3 Model-based fault detection

Driven by the increasing demand for higher safety and fast development of modern control theory, FDI has been studied extensively. Information redundancy is the basis of most fault detection methods. Generally, information redundancy can be classified into two main categories: Physical redundancy and Analytical redundancy, as shown in Figure \[72\].

The traditional physical redundancy requires the system equipped with redundant physical devices, and faces the problem of extra hardware costs and
additional weight/space to accommodate the equipment. Analytical redundancy uses the consistence between the variables/signals to check whether there is a fault. Analytical redundancy based FDI can be classified into three main approaches:

1. **signal-based approach**
   A fault can be detected by extracts symptoms from the signals. This method requires a priori knowledge about the relationship between signal symptoms and faults. Typical symptoms are magnitudes, means, covariances, amplitude envelope, correlation coefficients in time domain, or spectral power densities, spectrum in frequency domain, or combined time-frequency symptoms in wavelet domain. Signal-based approaches have been widely applied to mechanical engineering (e.g., vibration monitoring), electric motors (e.g., Motor Current Signature Analysis (MCSA)), etc. The signal-based approach is mainly designed for condition monitoring at the steady state.

2. **model-based approach**
   In this approach, a mathematical model (either quantitative or qualitative) is used to mimic the healthy process behaviour and the consistency between the model and the actual process is checked to detect faults. Under the condition that a good model has been obtained, the essential of this
approach is residual generation. The consistency is usually measured in terms of residuals. The residual, a function of time, is the discrepancy between the measured process variables against its estimates predicted by the mathematical model. Then, in an ideal situation, the residual is normally zero or close to zero when no component of the system fails, and becomes distinguishably different from zero when a fault occurs.

3. knowledge-base approach

In contrast to the model-based approach where a model is used, knowledge-based approach does not require a model. Instead, the information about the process is gained by analysing the history data of the process. This is known as the feature extraction, which can be carried out by some special 'intelligent' techniques (such as data mining, machine learning, pattern recognition, statistical learning, and so on). These features then are combined with some human knowledge and presented in the form of expert systems. This approach is suitable for some large system whose model is difficult to get, instead only a large amount of historical process data is available.

In the (quantitative) model-based fault detection that began in the early 1970s, three main approaches have been developed:

- system identification based parameter estimation approach [1, 2];
- parity relation approach [3, 4, 5, 6, 91];
- observer/filter-based approach [7, 8].

The model-based fault detection began at 1970s, and some of the typical work are [9], [10], [11], [12]. Since 1980s, the survey papers [7], [15], [16, 8], [14], [1, 2], [13] by Isermann, Gertler, Patton, Frank and Ding, respectively, have given a good overview of modern model-based fault detection methods. In 1999, a unified framework of model-based FDI was presented in a book [17] by Chen. Recently, Isermann [18] summarised the state of the art in the field of FDI.
2.3.1 Parameter estimation based fault detection

The parameter estimation approach is based directly on system identification techniques. The first attempts were made by [92] and [1]. In 1980s, Isermann presented his survey paper in parameter estimation based FDI approach [2]. Starting from this, a lot experiments and applications were reported. In 1990s, this approach was extended to on-line FDI by combining parameter identification and heuristic process knowledge [93], [13], and to nonlinear systems [94], time-varying systems. The latest development can be found in [18]. More comprehensive treatments of parameter estimation based fault detection are given by Isermann in [2, 13, 14, 64, 95]. More details on system identification can be found in Ljung’s book [68]. It should be pointed out that, the parameter estimation itself can be used to detect faults straightforward, it also paves the way for the other two model-based FDI approaches.

Recently, as a model-reference identification method, observer-based on-line parameter identification received more attention [96], [97], [42], [43]. In the high-gain observer based parameter estimation, one critical thing is the disturbance estimation delay (which is the delay between the disturbance and its estimate). One can apply time delay estimation (TDE) techniques [98] that has been the subject of many research efforts for a long time. Due to the nonlinearity of the time delay, the parameter identification is difficult. Moreover, its performance is highly depends on the SNR (signal-to-noise ratio). In the high-gain observer approach, because the actual disturbance is unmeasurable, it is the disturbance estimates to be used for parameter estimation. The estimation uncertainties involved in the observer makes the time delay estimation more difficult.

2.3.2 Observer-based fault detection

State observer has been widely used in many branches of science and engineering. According to the system characteristics, observers fall into two categories: Luenberger observers for deterministic systems [99], [10], [100] and Kalman filters for stochastic systems [101], [102], [17]. The former is limited to systems with accurate models neglecting both internal and external uncertainties; the latter considers external uncertainties as white noises [103], [102]. Unfortunately, modelling errors often lead to a poor performance in Kalman filtering [104], [105], [17], which makes Luenberger observer more popular in deterministic system research.
The heart of observer-based FDI is the generation of residuals, and the robustness has become the main theme in the last two decades. At the beginning, derived from Beard-Jones failure detection filter [10], [11], various fault detection filters were proposed [23], [30], [106], [107]. However, the robustness against model uncertainties was not considered in those fault detection filters. One common strategy for robust fault detection observer (RFDO) is to treat model uncertainties as some exoterical disturbances (unknown inputs) and de-couple them from the residual [17]. This makes the residual robust against unknown inputs (model uncertainties).

As the dominant approach for RFDO, the de-coupling approach has been developed quite well over the last two decades. The UIO was originally proposed in [20] and first employed to fault detection in [21]. The underlying principle is to make the state estimation error de-coupled from disturbances. Hence, the robust residual generation is achieved indirectly from those disturbance-free states. A lot of results of UIO-based RFDO have been reported [108], [17]. One assumption for UIOs is that the unknown input distribution matrix has been given a priori. Although this assumption can be satisfied in the case of exoterical disturbances, the distribution matrix for model uncertainties is generally unknown. This problem obstructed the application of UIO and addressed by using an estimated distribution matrix [25], [109], [17].

However, in FDI, it is not necessary to decouple the state estimation from disturbances, as the fault indication signal is the residual of output variables instead of the state variables. A direct approach is to de-couple the residual from disturbances in a straightforward manner. A common direct approach is eigenstructure assignment based UIO proposed in 1990s [25], [110], [24]. As a pole assignment method, Eigenstructure Assignment parametrises the feedback gain matrix with eigenvalues and a set of free parameters, and assigns the closed-loop poles to desired places arbitrarily [22], [111], [112]. It is worth to note that the non-unique solution to the gain matrix enables the optimal fault detection observer design. This is a main advantage of eigenstructure assignment.

The existence conditions for complete decoupling have been derived in the UIO approach [23], [21], and in the direct eigenstructure assignment approach by Patton [24], [25], respectively. The complete de-coupling, however, may not be possible, in some cases, due to the lack of design freedom. Moreover, it may be problematic because the fault effect may also be de-coupled. If the sufficient
condition of complete decoupling is not met, an approximate approach has to be taken. In this situation, the residual is not perfectly de-coupled from disturbances, but has a low sensitivity to disturbances and high sensitivity to faults. The basic concept is to measure the robustness and sensitivity by a suitable performance index, then optimise it. Thus, the RFDO design turns into an three-step optimisation problem: (1) performance index selection; (2) performance index parameterisation; (3) optimisation.

The performance of RFDO can be measured in time domain or in frequency domain. In the time domain, the criteria is expressed in terms of the so-called RMS (Root Mean Square) error or MSE (Mean Square Error) \[51\], \[26\], where 2-norm is adopted. The optimisation procedure is to minimise the MSE (or RMS) of fault-free residual sensitivity \( \sum r^2(t) |_{f=0} \) and maximise the MSE of fault residual sensitivity \( \sum r^2(t) |_{d=0} \). Simani adopted \( H_2 \)-norm in the time domain \[27\], Ding and Guo \[28\] proposed criteria in discrete time domain. An \( H_\infty \)-norm optimal index is also adopted \[113\], \[112\].

While most researchers proposed approaches in frequency domain and expressed the criteria in terms of transfer function matrix (TFM). Some examples are \( H_\infty/H_\infty \) norm \[29\], \( H_2/H_2 \) norm \[30\], \[8\], mixed \( H_2/H_\infty \) \[114\], \[115\], mixed \( H_-/H_\infty \) norm \[37\]. The \( H_2 \) observer minimises the \( H_2 \) norm of the residual under the assumption that the noises have known power spectral densities. \( H_\infty \) observers are good at dealing with deterministic bounded disturbances caused by model uncertainties, and some robustness performance can be guaranteed. The guaranteed performance, however, may be very conservative, as it is only optimised for the worst-case \[116\]. Similarly, \( H_- \) norm is used to enhance the effects of faults by maximising the minimum (singular) value of the fault transfer function matrix.

Once the performance index is selected, the next step is the parameterisation of the performance index. It is well known that the non-unique solution to the gain matrix enables the optimal fault detection observer design. Again, the eigenstructure assignment has attracted more attention in parameterisation, because both the observer gain matrix and the performance index (expressed in terms of TFMs) can be easily written in a certain eigenstructure form with a group of eigenvalues/poles and a set of free parameters \[117\], \[118\], \[22\]. Then, many iterative optimisation algorithms, such as gradient search \[119\], Genetic Algorithms \[110\], \[17\] are used to find the optimal gain matrix.
Meanwhile, stimulated by the $H_\infty$ control that firmly roots in the consideration of model uncertainties, the use of $H_\infty$ optimisation for robust residual generation has received more attention recently. Different from the approaches mentioned earlier (where the model uncertainties are tackled in the form of unknown inputs/disturbances), $H_\infty$ optimisation methods deal with the model uncertainties in a direct way. In this approach, the RFDO problem turns into a standard $H_\infty$ filtering problem \cite{119}, \cite{29}. A traditional $H_\infty$ optimisation approach is to solve the Algebraic Riccati Equation \cite{120}, \cite{121}. This optimal filtering problem can also be solved by means of a set of linear matrix inequations (LMIs) \cite{114} \cite{37}, \cite{38}.

However, the RFDO problem differs from the robust estimation/control problem, because the RFDO requires not only robustness against model uncertainties, but also sensitivity to faults. Hence, some trade-off between robustness and sensitivity has to be made. One usual way to compromise is to design the $H_\infty$ observer first, then check the fault sensitivity. If the fault sensitivity is too small, then one need to relax the robustness requirement and re-design the observer. This may be an iterative trial-and-error way. Some researchers have proposed to use $H_-$-norm to measure the fault sensitivity, rather than $H_\infty$ \cite{114}, \cite{115}, \cite{37}, \cite{34}. Another way is the use of frequency-dependent weighting functions \cite{116}.

\section{2.4 Dynamic observer design}

Although the observer/filter based fault detection theories have become rich, most of the existing methods are confined to the traditional (static) Luenberger structure and the main concern is the pole positions \cite{115}, \cite{17}, \cite{121}, \cite{22}, \cite{122}. Here, the term \textit{static observer} is used to denote the classic Luenberger observer, where a constant gain is used in the feedback path.

From the viewpoint of frequency domain, such a feedback gain does not have the ability of frequency shaping and puts some limitation on disturbance attenuation. It is of interest to change the observer structure and see how this would improve the fault detection performance. It is natural to introduce additional dynamics into observers. In order to distinguish from classic observers, the term \textit{dynamic observer} is used, in which a dynamic system is employed to replace the numerical gain matrix in the feedback path. Comparing to the static observers with only one gain matrix, dynamic observers provide more design freedom, and
presents both advantages and challenges.

Some preliminary work on dynamic observers has been done, but the attention has been mainly on the poles assignment. PIO (Proportional Integral Observer) and PMIO (Proportional Multiple Integral Observer) are discussed in [123], [124], [125], where the PIO is treated as a static observer with an additional integral term to deal with the steady state error. A discrete time PIO is proposed in [126], [127]. In [128], a dynamic observer design method is proposed as a dual of control design for state estimation. A similar work to this PhD study is the Lipschitz UIO [108] and input-output observer [122], where they still ignored the issue of zeros.

It is worth noting that, all the reports on dynamic observer design concerned on the poles and ignored the additional zeros introduced by the dynamic feedback gain. From the view of system performance, the performance depends not only on poles, but also on the position of zeros. Although the multivariable system zeros were first proposed by Rosenbrock over 30 years ago ([129]), the system zeros study received relatively less attention compared to the research on poles. More information on system zeros can be found in [130], [131], [132], [133].

Different from the reported work on dynamic filter design, this PhD research aims at establishing a zeros assignment approach in dynamic filter design and get systematic study on its dynamic behaviour. Here it is mainly the technique of zero assignment in dynamic observer to be investigated. To the best of our knowledge, there are no any results utilising zero assignment technique to design a fault detection observer.
Chapter 3

Nonlinear Least Squares for OE Model Identification

Generally, the study of a condition monitoring problem becomes simple if an accurate model of the system is given. In the on-board condition monitoring of gas turbine engines, a reduced order model is required due to the limited on-board computation resources. Furthermore, in order to retain the fault information in the residual, the model’s long term prediction (simulation) performance is of more interest, rather than the one-step ahead prediction. It is particularly true for detecting incipient faults. Although Output Error (OE) model shows a better simulation performance than other models (e.g., ARX, ARXMA, etc.), the dependency within the output errors presents challenges. This dependency makes LSE method biased leading to a poor long-term prediction. As described in a lot of literatures, even the model itself is linear, the objective function of OE model is highly nonlinear and some kind of iterative nonlinear optimisation is inevitable.

This chapter describes the modelling phase of this PhD work. The aim is to find a fast OE model identification algorithm for on-board condition monitoring which should fit the limited on-board computation facilities. This chapter starts with Section 3.1 discussing the performance criterion and structure selection. Section 3.2 defines objective function for OE modelling and presents the development of DNLS (Dynamic Nonlinear Least Squares) identification algorithm. In order to accelerate the identification speed of common NLS (Nonlinear Least Squares), two techniques are employed: (1) Iterative calculation of the gradient (the Jacobian). The dependency within output errors is taken into account by
adding an weighted sum of past gradients to the current Jacobian matrix. (2) Hessian approximation. In order to further accelerate the convergence speed at a relatively low computation cost, the second order Hessian matrix is used and approximated by the first order information. Finally, the potential of the modified NLS for reduced order OE modelling of gas turbine systems is illustrated in Section 3.3. Data gathered at an aero engine test-bed serves as the test vehicle to demonstrate the improvements in convergence speed and the reduction of computation costs.

### 3.1 Problem formulation

Consider a system corrupted by input noise $d_u$ and output noise $d_y$, as shown in Figure 3.1, where $d_u$ and $d_y$ denote either the measurement noise or the external disturbance. In the discrete time domain, the links between these variables can be described by

$$
\begin{align*}
  u(t) &= u^*(t) + d_u(t) \quad (3.1) \\
  y(t) &= y^*(t) + d_y(t) \quad (3.2)
\end{align*}
$$

and

$$
  y^*(t) = f(\theta, y^*(t-1), y^*(t-2), \ldots y^*(t-n), u^*(t-1), \ldots, u^*(t-m)). \quad (3.3)
$$

where $u^*$ and $y^*$ are the unmeasurable actual system input and output respectively, $\theta$ the parameter to be identified. The function $f(\cdot)$ is a mathematical description of how the system’s input/output variable relates to each other. $f(\cdot)$ can be linear or nonlinear depending on the physical nature of the plant.

A model is used to approximate the function $f(\cdot)$ by $\hat{f}(\cdot)$ and estimate the future output according the observed data sequence $u(t), y(t)$ up to time $t$. The system identification is such a technique to build the function structure and estimate the parameters $\theta$.

Not that the criterion on what is a good model is highly problem-dependent. It is useful to define and clarify what is a good model for condition monitoring, and select the right model structure for parameter identification. In this study, the objective is to find a ”good” reduced order model whose output error is robust to the disturbances $d_u, d_y$ and sensitive to the faults $f_a, f_s$. 
3.1.1 Criterion selection

In the initial stage, it was recognised that in the context of condition monitoring, the long-term prediction performance is more of interest than the one-step-ahead prediction. It is particularly true for detecting incipient faults. This subsection discusses why the long-term prediction performance is selected as the main criterion for condition monitoring.

Basically, there are two connection modes used for condition monitoring [64, 69]: (1) parallel connection for long-term prediction (which corresponds to the "Output Error" model or "Infinite Impulse Response" filter); (2) parallel-series connection for one-step ahead prediction (which corresponds to the "Equation Error" model);

In the parallel connection mode, as shown in Figure 3.1 (a), the model runs in parallel to the system (thereby, forming an so-called parallel model or IIR filter) and the error is termed as long-term prediction error, because the prediction \( \hat{y}(t) \) is calculated from the previous input \( u \) alone and has no direct link to the previous system output \( y \). This error is also called simulation error, as the model is often used to simulate the actual plant.

Figure 3.1 (b) shows a block diagram of the parallel-series connection mode, where the model consists two FIR (Finite Impulse Response) filters \( f_A \) and \( f_B \). \( f_A \) is the series part running in series to the plant, and \( f_B \) the series part running in parallel to the plant. Compared to the parallel connection mode, the obvious difference is the feeding back of actual system output \( y(t) \) to the model. Because \( \hat{y}(t) \) depends on the last system outputs \( y(t - 1), y(t - 2), \ldots \), it is called one-step-ahead prediction and the corresponding error is termed as one-step-ahead
prediction error, or simply prediction error.

![Graphs showing residuals of long-term prediction and one-step ahead prediction.](image)

Figure 3.2: Residuals of long-term prediction and one-step ahead prediction

The parallel-series model can represent the dynamic characteristics of the system to some extent, but it is not an exact duplication of the plant, because the plant output \( y(t - i) \) is used to correct the model prediction \( \hat{y}(t) \). Indeed, the parallel-series model is designed for fitting the output rather than simulating the plant. On the other hand, the parallel model repeats the dynamic behaviour of the plant in the same manner as the plant (that is to simulate the plant). Hence, it is easy to understand that the one-step-ahead predictor may not give the best fault detection performance, and the long-term prediction works better in terms of fault detection.
For illustrating this, both models are used to detect the actuator faults happening at 10 seconds. As shown in Figure 3.1.1 (a1), (b1), the abrupt and incipient faults are simulated and added to the input signal, respectively. Figure 3.1.1 (a2), (b2) shows the residuals given by the parallel-series model, where no obvious changes can be seen when faults happening. However, the magnitude change can be easily detected from the residual of the parallel model, as shown in Figure 3.1.1 (a3), (b3).

Therefore, in the context of fault detection, it is preferred to use the parallel connection rather than the parallel-series connection. As a consequence, the long-term prediction performance of parallel model is selected as the main criterion in this study.

### 3.1.2 Model selection: EE v.s. OE

From the viewpoint of system identification [69], there are two basic error concepts: (1) output error (2) equation error. Figure 3.3 gives an interpretation of these concepts in terms of a block diagram.

![Figure 3.3: Definitions of Output Error and Equation Error.](image)

**Equation Error (EE) Model**

The most simple way to describe the input-output relation $f(\cdot)$ is to represent it as a difference equation in the discrete time domain:

$$ y(t) = \varphi^T(t) \cdot \theta + e(t) $$

where

$$ \theta = [a_1 \ldots a_n a \ b_1 \ldots b_n]^T $$
is a parameter vector, and
\[ \varphi(t) = [y(t-1) \ldots y(t-n_a) u(t-1) \ldots u(t-n_b)]^T \] (3.6)
an observation vector at time \( t \), and \( e(t) \) the noise term acting as a direct error in
the difference equation. The model (3.4) is often called as equation error model.

By introducing the backward shift operator \( z^{-1} \), the model prediction can be
written in transfer function form:
\[ \hat{y}_{EE}(t) = [1 - A(z)]y(t) + B(z)u(t) \] (3.7)
where
\[ A(z) = 1 + a_1z^{-1} + \ldots + a_{n_a}z^{-n_a} \] (3.8)
and
\[ B(z) = b_1z^{-1} + \ldots + a_{n_b}z^{-n_b}. \] (3.9)

Alternatively, it can be written in the vector form
\[ \hat{y}_{EE}(t) = \varphi^T(t) \cdot \theta \] (3.10)
And the quantities
\[ r_{EE}(t) = y(t) - \hat{y}_{EE}(t) = y(t) - \varphi^T(t) \cdot \theta \] (3.11)
are called equation errors.

Recall the parallel-series model (see Figure 3.1), one can see that the EE model
in fact is some kind of parallel-series model. \( 1 - A(z) \) corresponds to the series
part \( f_A \), \( B(z) \) is the counterpart of \( f_B \) and \( \hat{y}_{EE}(t) \) in (3.7) is the one-step
ahead prediction.

The model (3.4) is also called an ARX (AutoRegressive eXogeneous) model,
where AR refers to the autoregressive part \( A(z)y(t) \), \( X \) to the extra input \( B(z)u(t) \),
and \( \varphi(t) \) to the regression vector at time \( t \).

**Output Error (OE) Model**
The EE model separates the relation between the input and output into two
transfer functions \( 1 - A(z) \) (relating \( \hat{y} \) to \( y \)) and \( B(z) \) (relating \( \hat{y} \) to \( u \)). From a
physical point of view, it may seem more natural to present \( f() \) in one transfer function. An immediate way of presenting a transfer function is to parameterise \( f() \) as a rational function. Thus the model is given as

\[
\hat{y}_{OE}(t) = \frac{B(z)}{F(z)}u(t)
\]  

(3.12)

with

\[
B(z) = b_1 z^{-1} + \ldots + a_{nb} z^{-nb}
\]  

(3.13)

and

\[
F(z) = 1 + f_1 z^{-1} + \ldots + f_{nf} z^{-nf}
\]  

(3.14)

The parameter vectors to be identified is

\[
\theta = [f_1 \ f_2 \ldots f_{nf} \ b_1 \ldots b_{nb}]^T
\]  

(3.15)

By multiplying both side of (3.12) with \( F(z) \) and moving terms \( f_i z^{-i}\hat{y}(t), (i = 1, \ldots, n_f) \) to the right side, the prediction \( \hat{y}_{OE}(t) \) can be rewritten in the vector form [68, 67]:

\[
\hat{y}_{OE}(t) = \hat{\varphi}^T(t) \cdot \theta
\]  

(3.16)

with pseudo-regression vector

\[
\hat{\varphi}(t) = [\hat{y}(t-1) \ldots \hat{y}(t-n_f) \ u(t-1) \ldots u(t-n_b)]^T
\]  

(3.17)

The prediction error of (3.16)

\[
r_{OE}(t) = y(t) - \hat{y}_{OE}(t) = y(t) - \hat{\varphi}^T(t) \cdot \theta
\]  

(3.18)

is called Output Error.

It is worth to note that the pseudo-regression vector (3.17) in OE model differs from the regressor (3.6) in EE model. Compared to \( y(t-j), j = 1,2,\ldots,n_a \) in the regressor (3.6) (which are the measured plant outputs), \( \hat{y}(t-i) \) in the pseudo-regression vector (3.17) are not observed. Instead, \( \hat{y}(t-i) \) are the previous model predictions.

Remark 3.1. One can find that the model connections are closely linked with the error concepts. The parallel connection corresponds to the output error, and the
parallel-series connection is associated with the equation error.

As discussed earlier, a good model for condition monitoring is the parallel model, and a good criterion is the long-term prediction performance. It follows that OE model is better than EE model in terms of condition monitoring, and the next section will solve the parameter identification of OE model.

### 3.2 NLS for OE parameter identification

The least-squares estimate (LSE) algorithm has been proved to be very effective for the ARX model identification. The objective of LSE is to minimise the mean squared equation error (MSEE):

\[
V_{EE}(\theta) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} [y(t) - \varphi^T(t)\theta]^2
\]

where \( V_{EE}(\theta) \) is a quadratic function of the parameter \( \theta \). It has been proved in literatures that the LSE can be successfully applied to the ARX model and gives the best parameters estimation in terms of minimising \( V_{EE}(\theta) \). However, the estimates of \( \{a_i\} \) given by LSE may be biased if the residuals are correlated ([69], [66], p.207, p.256 in [68], [70]).

Because of the dependence existing within the residual, the LSE method fails to give unbiased parameter estimates resulting in a relatively poor long-term prediction performance. It is of interest to note that: the biased parameter estimates benefit to the smaller one-step prediction error. This agree with the point that LSE aims to minimise the one-step-ahead prediction error (3.19).

On the other hand, the objective of identifying OE model is to minimise the mean squared output error (MSOE):

\[
V_{OE}(\theta) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} [y(t) - \hat{\varphi}^T(t)\theta]^2
\]

Note that, although they have the similar form in appearance, \( V_{OE}(\theta) \) differs \( V_{EE}(\theta) \) a lot due to the difference between \( \varphi(t) \) (3.6) and \( \hat{\varphi}(t) \) (3.17). Hence, (3.19) is a quadratic function, but (3.20) is highly nonlinear with respect to the parameter \( \theta \) [67], [46].
Because of the high nonlinearity of OE objective function, the OE model identification is not an easy task, even if the model is linear. Some iterative optimisation algorithm is inevitable, where the identification algorithm can be expressed as

$$
\theta_{k+1} = \theta_k - \eta \cdot [R(k)]^{-1} \cdot \frac{\partial E(\theta_k)}{\partial \theta}
$$

(3.21)

where \( \eta \) is a series of positive scalars (in term of step size) tending to zero or small value, and \( R(k) \) a \( n \)-by-\( n \) (\( n = \text{dim } \theta \)) positive definite matrix to modify the search direction \(-\frac{\partial E(\theta_k)}{\partial \theta}\). Here, \( \frac{\partial E(\theta_k)}{\partial \theta} \) denotes the derivatives of \( E(\theta) \) with respect to \( \theta \) at \( k \)-th iteration.

The following sections will give the technique details of the proposed fast identification algorithm. For the sake of compact notation, the terms \( r(t), V(\theta) \) will denote the output error \( r_{OE}(t) \), the OE objective function \( V_{OE}(\theta) \), respectively, in the following sections otherwise specified.

### 3.2.1 Calculation of \( \frac{\partial V(\theta_k)}{\partial \theta} \) and the Jacobian

It is important to keep in mind that \( \hat{\phi}(t) \) in (3.20) involves model prediction \( \hat{y}(t-i), i = 1 \cdots n_f \), and \( \hat{y}(t-i) \) is a function of parameters \( \theta \). It yields that \( \hat{\phi}(t) \) depends on \( \theta \) too. For explicitly expressing the links between \( \hat{y} \) and \( \theta \), \( \hat{\phi}(t, \theta_k) \) is adopted to denote \( \hat{\phi}(t) \) at \( k \)-th optimisation step:

$$
\hat{\phi}(t, \theta_k) = [\hat{y}(t-1, \theta_k) \ldots \hat{y}(t-n_f, \theta_k) \ u(t-1) \ldots u(t-n_b)]^T
$$

(3.22)

Let \( g_k(t) \) denote the individual local gradient information at time \( t \) during the \( k \)-th iteration, that is

$$
g_k(t) = \frac{\partial r(t)}{\partial \theta},
$$

(3.23)

followed by

$$
g_k(t) = \frac{\partial (y(t) - \hat{y}(t, \theta_k))}{\partial \theta} = -\frac{\partial (\hat{y}(t, \theta_k))}{\partial \theta}.
$$

(3.24)

Since

$$
\frac{\partial (\hat{y}(t, \theta_k))}{\partial \theta} = \frac{\partial (\hat{\phi}^T(t, \theta_k) \cdot \theta_k)}{\partial \theta} = (\hat{\phi}(t, \theta_k) + \frac{\partial \hat{\phi}(t, \theta_k)}{\partial \theta} \cdot \theta_k),
$$

(3.25)
it follows that
\[ g_k(t) = -[\hat{\phi}(t, \theta_k) + \frac{\partial \hat{\phi}(t, \theta_k)}{\partial \theta} \cdot \theta_k]. \] (3.26)

According to the definition of \( \hat{\phi}(t, \theta_k) \) and (3.24), \( \frac{\partial \hat{\phi}(t, \theta_k)}{\partial \theta} \) is
\[
\begin{align*}
\frac{\partial \hat{\phi}(t, \theta_k)}{\partial \theta} &= \left[ \frac{\partial \hat{y}(t-1, \theta_k)}{\partial \theta} \cdots \frac{\partial \hat{y}(t-n_f, \theta_k)}{\partial \theta} \right] \cdot \left[ \frac{\partial u(t-1)}{\partial \theta} \cdots \frac{\partial u(t-n_b)}{\partial \theta} \right] \\
&= -[g_k(t-1) \cdots g_k(t-n_f) 0 \cdots 0]
\end{align*}
\] (3.27)

Substituting (3.27) into (3.26) gives
\[ g_k(t) = -[\hat{\phi}(t, \theta_k) - [g_k(t-1) \cdots g_k(t-n_f) 0 \cdots 0] \cdot \theta_k] \] (3.28)

Equation (3.28) indicates that \( g_k(t) \) at time instant \( t \) not only depends on current observation vector \( \hat{\phi}(t, \theta) \) but also depends on the previous gradients \( \{g_k(\tau)\}, (\tau < t) \). Therefore, the iterative calculation of \( g_k(t) \) (3.28) represents the dependency within prediction errors \( \{r(t)\} \).

The gradient on all the observed data is calculated by summing all values of local gradients.
\[
\frac{\partial V(\theta_k)}{\partial \theta} = \frac{1}{2} \sum_{t=1}^{N} \frac{\partial r^2(t)}{\partial \theta} = \sum_{t=1}^{N} r(t) \cdot g_k(t) 
\] (3.29)

Compared with ARX model (where the gradient is \( \phi(t) \) alone), the calculation of gradient as (3.28) differs at the additional sum terms \([g_k(t-1) \cdots g_k(t-n_f) 0 \cdots 0]\) \( \cdot \theta_k \). The added terms enable the derivative of \( V(\theta) \) reflect the actual gradient of objective function (3.20) correctly.

Note that \( g_k(t) \) is calculated at every time \( t \). As a result, the Jacobian of objective function (3.20) is achieved simply by transforming the sequence \( \{g_k(t)\} \) into a matrix form:
\[
J = [g_k(1) \ g_k(2) \cdots \ g_k(N)]^T
\] (3.30)
where \( J \) is \( N \)-by-(\( n_f + n_b \)) matrix.

Let a column vector \( \Gamma \) to denote the long-term prediction error sequence \( \Gamma = [r(1) \ r(2) \cdots \ r(N)]^T \), then (3.29) can be rewritten as
\[
\frac{\partial V(\theta_k)}{\partial \theta} = J^T \cdot \Gamma
\] (3.31)
3.2.2 Approximation of $R(k)$ and the Hessian

As shown in many literatures, for minimising a nonlinear function, it is may be inefficient to use the first-order alone as the search direction. In most cases, the inverse of gradient does not point to the minimum point straightaway. It is particularly true when the objective function surface has a valley, e.g., the Rosenbrock function. Thus a matrix $R(k)$ is employed to adjust the optimisation direction from $\frac{\partial E(\theta)}{\partial \theta}$. With the aid of Taylor expansion, it has been shown that using a quadratic model to approximate the high nonlinear function (3.20) is beneficial for improving the optimisation convergence. A good selection of $R(k)$ in (3.21) is the Hessian matrix. However, the calculation of the Hessian matrix is computation-consuming. In order to reduce the computation costs for the on-board application, the Hessian need to be approximated properly at a lower cost.

One of the solution is to make use of the Jacobian to approximate the Hessian. Consider the second-order Taylor expansion of function (3.20) around some parameter $\theta^*$

$$E(\theta) \approx E(\theta^*) + \left[ \frac{\partial E(\theta)}{\partial \theta} \bigg| _{\theta=\theta^*} \right]^T (\theta - \theta^*) + \frac{1}{2} (\theta - \theta^*)^T H(\theta^*)(\theta - \theta^*)$$

(3.32)

where $H(\theta^*)$ is the Hessian matrix evaluated at $\theta^*$. Note that this approximation is valid when $\theta$ is in the neighborhood of $\theta^*$.

For compact notation, $\frac{\partial E(\theta)}{\partial \theta} | _{\theta=\theta^*}$ is replaced by $\frac{\partial E(\theta^*)}{\partial \theta}$. From expansion (3.32), the condition of minimum of $E(\theta)$ can be expressed as

$$\frac{\partial E(\theta^*)}{\partial \theta} + H(\theta^*)(\theta - \theta^*) \approx 0$$

(3.33)

followed by

$$\theta \approx \theta^* - H(\theta^*)^{-1} \cdot \frac{\partial E(\theta^*)}{\partial \theta}$$

(3.34)

Comparing (3.34) with (3.21), one can easily find that the search direction should be modified by the inverse Hessian matrix. It follows that the $R(k)$ can be replaced by the Hessian matrix.

The problem turns into an approximation of the Hessian matrix. According to Gauss-Newton methods, an easy way to form Hessian estimate is to make use
of the first derivative information, as shown below.

$$
H(\theta) \triangleq \frac{\partial^2 V(\theta)}{\partial \theta \partial \theta^T} \\
= \sum_{t=1}^{N} \frac{\partial^2 (r(t))}{\partial \theta \partial \theta^T} \\
= \sum_{t=1}^{N} \left[ \frac{\partial r(t)}{\partial \theta} \cdot \frac{\partial r(t)}{\partial \theta^T} + r(t) \frac{\partial^2 r(t)}{\partial \theta \partial \theta^T} \right] \\
= J^T \cdot J + r(t) \frac{\partial^2 r(t)}{\partial \theta \partial \theta^T} 
$$

(3.35)

It is therefore convenient to ignore the second term on the right-hand side and approximate $H(\theta)$ by $J^T \cdot J$. It follows that

$$
R(k) = J^T \cdot J 
$$

(3.36)

In terms of the identification convergence, this approximation approach potentially offer the best trade-off of two worlds: First, by using the Hessian matrix as $R(k)$, the search direction is modified from inverse gradient direction to point toward to the minimum more straightaway. Secondly, it only needs to compute the first-order derivatives that have been available in the computation of local gradients $g_k(t)$. Thus, the calculation of the Hessian does not introduce too much more extra computation burden, and results a better parameter updating direction.

As a result, the improved NLS algorithm for reduced order OE model is named Dynamic Nonlinear Least Squares (DNLS) and given as follows:

**Algorithm 3.1** (Dynamic Nonlinear Least Squares algorithm for OE model identification).

1) Let $k = 0$, $g_k(1) \ldots g_k(n_f) = 0$, and set the initial values of $\theta_k$ according to a priori knowledge;

2) At each time instant $t$, $(t = n_f + 1, \ldots, N)$, compute the local gradient $g_k(t)$ by

$$
g_k(t) = -[\hat{\phi}(t, \theta_k) - [g_k(t - 1) \ldots g_k(t - n_f) 0 \ldots 0] \cdot \theta_k];
$$

3) At time instant $N$, rearrange $\{g_k(t)\}$ to form the Jacobian $J = [g_k(1) g_k(2) \ldots g_k(N)]^T$

4) Update the parameter by $\theta_{k+1} = \theta_k - \eta \cdot (J^T \cdot J)^{-1} \cdot J^T \cdot \Gamma$, where $\Gamma = [r(1) r(2) \ldots r(N)]^T$ and $\eta$ is a fixed or adjustable step-size.
5) Stop condition check: check whether the maximum iteration has been achieved or the updating of \( \theta_k \) has been very small. If not, set \( k = k + 1 \) and go back Step 2.

Remark 3.2. Since the Hessian matrix is approximated by \( J^T \cdot J \), the approximation is only accurate enough when the parameters \( \theta \) is close enough to the minimum, as shown in the second-order Taylor expansion \( (3.32) \). In this case, the step size \( \eta \) is equal to 1. However, in the earlier stage of search where \( \theta \) may be far from the minimum, \( \eta \) is not equal to 1 and a line search method is adopted to select the optimal value of \( \eta \) in each search iteration.

Remark 3.3. The dependency within long-term prediction errors is solved by computing the gradient in an iterative manner, as shown in \( (3.28) \), leading to a better gradient calculation.

Remark 3.4. In the proposed DNLS, the objective function is approximated by a quadratic function that is more accurate than the first-order approximation, thus the search direction is better than those common deepest descent methods. Furthermore, the Jacobian \( J \) is just a rearrangement of sequence of gradients \( g(t) \) \( (t = 1 \cdots N) \) without additional computation and the approximation of the Hessian does not need more computation than common deepest descent methods. Therefore, one of the benefits is that the improvement on performance is only paid by a small increase of computation expense, which make this method is suitable for on-board modelling and condition monitoring.

3.3 Application and results

In this section, Algorithm 3.1 is employed to identify the parameters of the reduced order OE model of a two shaft gas turbine engine. Real engine data gathered from normal engine operation at the engine test-bed are used. In the duration of this test, the angle of the VGVs (Variable Guide Vanes) of the low pressure compressor and the reheat nozzle area were fixed to their low speed positions, the engine fuel flow \( W_f(t) \) is the control input and the high pressure shaft speed \( N_{HP}(t) \) is the primary output. Because the engine runs at the low speed operating point, the input/output data are first preprocessed by subtracting the physical equilibrium. Therefore, the data used indeed for dynamic modelling are \( \Delta W_f(t) \) and \( \Delta N_{HP}(t) \). For simplicity of notation, let \( u(t), y(t) \) denote the input
\( \Delta W_f(t) \) and output \( \Delta N_{HP}(t) \), respectively. 1500 data pairs were collected in total. The first 750 pairs compose the training data set for parameters estimation and the remaining 750 pairs make up the validation data set for model validation.

Figure 3.4 illustrates the distribution of the training data set and validation data set, where blue ‘*’ represents the training data and red ‘○’ the test data. It is obvious that these two data sets cover a little different dynamics of the gas turbine, although they are overlapped around the centre which represents the current operating point.

![Distribution of training data set and validation data set](image.png)

Figure 3.4: Scatter plot of the output \( y(t) \) versus the input \( u(t - 1) \).

The aim of dynamic modelling is to obtain a reduced order model whose long-term prediction errors are minimised. To measure the algorithm performances, two criteria are examined: (1) prediction errors and (2) computation costs. The prediction accuracy is measured by MSOE (Mean Square Output Error)

\[
MSOE = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} [y(t) - \hat{y}_{OE}(t)]
\]  
(3.37)

where \( N = 750 \) is the length of the training or validation data set. Note that some methods (e.g., LSE) are for minimising one-step ahead prediction errors. For such algorithms, the MSEE (Mean Square Equation Error) on training data set is used for comparison.

\[
MSEE = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} [y(t) - \hat{y}_{EE}(t)].
\]  
(3.38)
The standard deviation of the errors is also adopted

\[
STD = \sqrt{\frac{1}{N-1} \sum_{t=1}^{N} (r(t) - E(r(t)))^2}
\]  

where \(r(t)\) is either the OE errors or EE errors, \(E\) denotes the mathematical expectation.

The computation cost is measured by how many evaluations of the objective function are carried out during the identification. Because the main computation burden in most iterative optimisation algorithms is the calculation of the long-term prediction sequence \(\hat{y}_{OE}(t), t = 1 \cdots N\).

### 3.3.1 First-order model

Here, the first order model of the gas turbine engine is considered, where \(\phi(t) = [\hat{y}(t-1), u(t-1)]^T\) and \(\theta = [f_1, b_1]^T\). In these experiments, although the model is a linear model, the objective function (3.20) is a high order nonlinear surface with a narrow valley around the minimum point as shown in Figure 3.5.

![Objective Function of Long-term Prediction](image)

**Figure 3.5:** Objective function of the first-order model

For comparison, five different algorithms are presented here. The LSE and ARX approaches provided by *System Identification toolbox* do not contain iterative search, hence their computation costs are set 1. All the rest algorithms are iterative search methods. The standard OE method and the RIV (Refined Instrumental Variable) [134] [135] are also included for comparison. A fuzzy neural
Table 3.1: Comparison on training data set (1st order model)

<table>
<thead>
<tr>
<th>Methods</th>
<th>Computation Costs</th>
<th>1-step-ahead prediction</th>
<th>Long-term prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>STD</td>
<td>MSEE</td>
</tr>
<tr>
<td>LSE</td>
<td>1</td>
<td>0.46133</td>
<td>0.21284</td>
</tr>
<tr>
<td>ARX</td>
<td>1</td>
<td>0.46169</td>
<td>0.21322</td>
</tr>
<tr>
<td>OE</td>
<td>N/A</td>
<td>0.54868</td>
<td>0.30120</td>
</tr>
<tr>
<td>RIV</td>
<td>N/A</td>
<td>0.54713</td>
<td>0.29965</td>
</tr>
<tr>
<td>ANFIS</td>
<td>250</td>
<td>0.43276</td>
<td>0.18728</td>
</tr>
<tr>
<td>Exhaustive Search</td>
<td>12000</td>
<td>0.48241</td>
<td>0.23280</td>
</tr>
<tr>
<td>Gradient Descent(a)</td>
<td>1000</td>
<td>0.48270</td>
<td>0.23312</td>
</tr>
<tr>
<td>DGD(b)</td>
<td>101</td>
<td>0.48239</td>
<td>0.23281</td>
</tr>
<tr>
<td>DNLS1 (c)</td>
<td>59</td>
<td>0.48238</td>
<td>0.23280</td>
</tr>
<tr>
<td>DNLS2 (c)</td>
<td>80</td>
<td>0.48237</td>
<td>0.23280</td>
</tr>
</tbody>
</table>

\(a\)The gradient search approach uses inverse gradient as search direction  
\(b\)The DGD approach uses BFGS algorithm to adjust search direction  
\(c\)DNLS1 and DNLS2 stops after 59 and 80 objective function evaluations respectively because of different stop conditions

networks ANFIS (Adaptive Networks-based Fuzzy Inference System) is also used. The results show that such neural networks are not suitable for long-term prediction.

The exhaustive search method examines each possible parameter value in order to find the 'possible' global solution. In our experiments, it runs in two main steps and the total number of objective evaluation is 12000.

In all the gradient-based iterative algorithms, \([f_1, b_1] = [0.5 0.05]\) is set as the default initial values of parameters and the initial step size is 0.1. The gradient search method uses the inverse gradient as search direction and set \(R(k)\) in (3.21) as a identity matrix. In DGD (Dynamic Gradient Descent) approach\([65]\), the BFGS (Broyden-Fletcher-Goldfarb-Shanno)\([68]\) are adopted to modify the search direction from inverse gradient direction. The detailed search progress of DGD is presented in Figure 3.6. The DGD algorithm needs about 100 objective function evaluations to approach to the optimum point, and a distance from the minimum still can be seen. One reason is that the search direction does not point to the minimum at the beginning stage. Further reason is that the search route continually zigzags from one side of the valley to another after entering the valley as shown in Figure 3.6.

In the proposed DNLS approaches, because of line search, two different stop
The DGD algorithm: (a) Searching route of DGD, (b) Training errors and (c) step size.

conditions are tested in DNLS1 and DNLS2, respectively. The search process of DNLS1 is shown in Figure 3.7. Parentally, the solution point given by the DNLS is closer to the minimum and the convergence speed is accelerated, as depicted in Figure 3.7. The search direction of DNLS points a more straightforward way to the minimum. Even in the valley, it still looks better. More clearly, from Figure 3.7 (b), (c), it can be seen that only 7 steps are involved to arrive the optimum point and the value of objective function drops dramatically and steadily. Compared to DGD, this improvement is benefited from a better Hessian approximation in (3.36). Figure 3.8 shows the DNLS results of long-term prediction on validation data with MSOE of 9.131786.

Figure 3.7: The DNLS algorithm: (a) Searching route, (b) training errors and (c) training stepsize.

The comparison of different methods is shown in Table 3.1 (on training data)
and Table 3.2 (on validation data). It can be concluded from these tables that:
(1) In terms of one-step-ahead prediction (equation error), LSE, ANFIS, ARX achieved slightly smaller equation errors than those methods for long-term prediction. However, LSE, ANFIS, ARX failed on long-term prediction on both training data set and validation data set; (2) In terms of long-term prediction (output error), the methods OE, RIV, exhaustive search, Gradient Descent, DGD and DNLS obtain similar results. Their performances on long-term prediction are better than those methods designed for one-step-prediction. (3) In terms of computation costs, the DNLS is advantageous over the exhaustive search, Gradient Descent, DGD approaches. The main contribution of DNLS is the reduction of computation costs that makes this approach is better for on-board identification.

One thing interesting is that the neural network ANFIS achieves the smallest MSEE 0.18728 on training data, but it gives the worst MSOE performance on both training data and validation data. Although static feedforward neural networks (such as ANFIS) have good ability in approximating static mathematical functions, they may be not suitable for dynamic system simulation (in terms of long-term prediction). The smallest MSEE given by the ANFIS on the training data set may be caused by the overfit during the training stage. Hence, the ANFIS interpolates the training data set very well, but it fails to extrapolate the validation data.
Table 3.2: Comparison on validation data set (1st order model)

<table>
<thead>
<tr>
<th>Methods</th>
<th>STD</th>
<th>MSOE</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSE</td>
<td>3.7796</td>
<td>16.8985</td>
</tr>
<tr>
<td>ARX</td>
<td>4.0895</td>
<td>19.5996</td>
</tr>
<tr>
<td>OE</td>
<td>2.9301</td>
<td>10.2658</td>
</tr>
<tr>
<td>RIV</td>
<td>3.2490</td>
<td>12.2963</td>
</tr>
<tr>
<td>ANFIS</td>
<td>$3.0258 \times 10^{3}$</td>
<td>$1.1908 \times 10^{7}$</td>
</tr>
<tr>
<td>Exhaustive Search</td>
<td>3.1807</td>
<td>11.8369</td>
</tr>
<tr>
<td>Gradient Descent</td>
<td>8.1806</td>
<td>11.8311</td>
</tr>
<tr>
<td>DGD</td>
<td>3.1807</td>
<td>11.8369</td>
</tr>
<tr>
<td>DNLS1</td>
<td>3.1811</td>
<td>11.8398</td>
</tr>
<tr>
<td>DNLS2</td>
<td>3.1807</td>
<td>11.8368</td>
</tr>
</tbody>
</table>

3.3.2 Second-order model

The second-order model also is identified to test the ability of DNLS, where 

\[
\hat{\phi}(t) = [\hat{y}(t-1) \ \hat{y}(t-2) \ u(t-1)u(t-2)]^T \quad \text{and} \quad \theta = [f_1 \ f_2 \ b_1 \ b_2]^T.
\]

The MSE drops to 3.31867176 and the computation cost is 60, as shown in Figure 3.9. The parameters of such a model are 

\[
\theta = [1.8604765 \ -0.8641699 \ 0.07045335 \ -0.007475994]^T.
\]

The results of the second-order model demonstrate that the algorithm is still very effective for high order models.

Figure 3.9: Long-term prediction of DNLS for the second-order model on the validation data.

The comparison against other approaches is listed in Table 3.3. It can be seen
that the computation cost of the proposed DNLS is just 103.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Costs</th>
<th>Training Data</th>
<th></th>
<th></th>
<th>Validation Data</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>STD</td>
<td>MSOE</td>
<td>STD</td>
<td>MSOE</td>
<td>STD</td>
<td>MSOE</td>
</tr>
<tr>
<td>LSE</td>
<td>1</td>
<td>3.41291</td>
<td>1.16613*</td>
<td>2.87529</td>
<td>10.3296</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ARX</td>
<td>1</td>
<td>3.93632</td>
<td>1.55375*</td>
<td>3.05683</td>
<td>11.5071</td>
<td></td>
<td></td>
</tr>
<tr>
<td>OE</td>
<td>N/A</td>
<td>1.32140</td>
<td>1.89871</td>
<td>1.85980</td>
<td>4.76809</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RIV</td>
<td>N/A</td>
<td>1.25340</td>
<td>1.81345</td>
<td>1.97333</td>
<td>5.19088</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exhaustive Search</td>
<td>10^6</td>
<td>1.25624</td>
<td>1.78773</td>
<td>1.95562</td>
<td>5.16392</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gradient Descent</td>
<td>20000</td>
<td>2.30235</td>
<td>5.39615</td>
<td>2.74971</td>
<td>9.22707</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DGD</td>
<td>1000</td>
<td>1.25625</td>
<td>1.78773</td>
<td>1.95560</td>
<td>5.16380</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DNLS</td>
<td>103</td>
<td>1.25624</td>
<td>1.78773</td>
<td>1.95563</td>
<td>5.16391</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*These two are MSEEs, because LSE, ARX are for EE model only.

3.4 Summary

In this chapter, by a comparative study, the OE model is selected as the model for modelling gas turbine engines in terms of long-term prediction. The discussion on parallel/serial model connection and the corresponding equation/output error shed light on the way to unbiased parameter identification. It has been proved that LSE gives biased parameter estimation in the context of reduced order modelling. In the identification algorithm design, the nonlinear least-squares method is used and problem of correlated prediction errors in the OE model is resolved by calculating the gradient in an iterative manner. In order to accelerate the identification speed and meet the limitation on the on-board computation resources, the second-order Hessian matrix is approximated by the first-order Jacobian matrix.

The main contribution of this study is to propose an iterative calculation of the gradient to deal with the correlated prediction errors. Furthermore, by approximating the Hessian with the the Jacobian, the identification speed is accelerated with a minor increase of computation costs. The experiment results on modelling a gas turbine engine have shown that the proposed approach achieves a faster convergence speed at relatively lower computational costs. improved significantly.
Chapter 4

High-gain Observer-based Parameter Estimation

This chapter focuses on parameter estimation in dynamic systems with bounded time-varying parameters, so that a fault in the plant can be detected by checking the parameter variation range. A high gain observer and finite state machine based identification method is presented in this chapter. As a model reference identification technique, the observer technique is used for adaptive parameter estimation. One of these, an interesting high-gain observer has been proposed by Gao etc. [42], [43] to identify model parameter perturbations, where both the unmeasured states and disturbances are estimated and the variation of parameters is identified within an ARX framework. The high-gain observer shows its ability to identify the model parameter perturbations under the corruption of bounded process and measurement noises.

This chapter commences with Section 4.1 introducing the dynamic system with parameter variation, followed by the design of high-gain observer for disturbance estimation in Section 4.2. In Section 4.3, based on the estimated states and disturbances, the parameter variation can be identified within an ARX framework. Section 4.4 shows a FSM-based (Finite State Machine) adaptive change detection algorithm for monitoring the parameter variation. As illustrated in Section 4.5, simulation results on a gas turbine engine show the performance of this approach. It is worth to note that, in this chapter, the observer has to be designed properly so that no time delay exists between the disturbance and its estimate. In the next chapter, this constraint will be removed by using filter approximation techniques.
4.1 Introduction

Consider a system corrupted by bounded input/output noise

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + \omega_i(t) \\
y(t) &= Cx(t) + \omega_o(t)
\end{align*}
\] (4.1)

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^p\) the input, \(y \in \mathbb{R}^r\) the output, \(\omega_i(t) \in \mathbb{R}^n\) and \(\omega_o(t) \in \mathbb{R}^r\) are bounded input and output noise vectors, respectively, and \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{r \times n}\).

Assume that, due to the system degradation or faults, the parameters of matrices \(A\) and \(B\) may vary as follows:

\[
\begin{align*}
A &= A_0 + \Delta A \\
B &= B_0 + \Delta B
\end{align*}
\] (4.2)

where \(A_0, B_0\) are known as matrices of the nominal model (corresponds to a healthy plant), system (4.1) can be written as

\[
\begin{align*}
\dot{x}(t) &= A_0x(t) + B_0u(t) + B_d d(t) + \omega_i(t) \\
y(t) &= Cx(t) + \omega_o(t)
\end{align*}
\] (4.3)

where

\(d(t) = \Delta A x(t) + \Delta B u(t)\) (4.4)

and

\(B_d = I_n\) (4.5)

The aim of time-varying parameter estimation is to estimate the model uncertainties \(\Delta A, \Delta B\) from the measured inputs/outputs. It can be seen that the disturbance model (4.4) is a (static) 2-input-1-output ARX model without time delay, where \(x(t)\) and \(u(t)\) are regarded as the inputs and \(d(t)\) the output. If \(u(t), x(t)\) and \(d(t)\) are known, \([\Delta A, \Delta B]\) can be identified straight away. The basic idea of the proposed method is to obtain an estimate \(\hat{x}\) of the plant state \(x\) and \(\hat{d}\) of the disturbance \(d\) simultaneously by using the observer technique, and these estimates are then used as substitutes for \(x, d\). In theory, if \(\hat{x}, \hat{d}\) can be made as same as the actual \(x\) and \(d\) exactly, a good estimate of \([\Delta A, \Delta B]\) is
reachable. The state estimation is standard. Thus, the challenge lies at estimating the unmeasurable disturbance $d(t)$. The main benefit of the proposed method is the identification of $[\Delta A, \Delta B]$ will be insensitive to the exoterical disturbances $\omega_i, \omega_o$.

4.2 High-gain observer for disturbance estimation

In order to estimate the disturbance as well as the states, a high-gain observer is adopted and modified here for estimating the disturbance. Denote

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ d(t) \\ w_o(t) \end{bmatrix} \in \mathbb{R}^{2n+r}$$

and

$$\bar{A} = \begin{bmatrix} A_0 & B_d & 0 \\ 0 & 0 & -I_r \\ 0 & 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} B_0 \\ 0_{n\times p} \\ 0_{r\times p} \end{bmatrix},$$

$$\bar{E} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0_{r\times n} \end{bmatrix}, \bar{C} = \begin{bmatrix} C & 0 & I_r \end{bmatrix},$$

$$\bar{G} = \begin{bmatrix} I_n \\ 0_{n\times n} \\ 0_{r\times n} \end{bmatrix}, \bar{H} = \begin{bmatrix} 0_{n\times n} \\ I_{n\times n} \\ 0_{r\times n} \end{bmatrix}, \bar{N} = \begin{bmatrix} 0_{n\times r} \\ 0_{n\times r} \\ I_r \end{bmatrix}.$$  

(4.7)

an augmented descriptor system can be obtained from (4.3) and (4.7) to give

$$\begin{align*}
\bar{E} \dot{x}(t) &= \bar{A} x(t) + \bar{B} u(t) + \bar{G} \omega_i(t) + \bar{H} \dot{d}(t) - \bar{N} \omega_o(t) \\
y(t) &= \bar{C} \bar{x}(t).
\end{align*}$$

(4.8)

In this study, $\dot{d}(t)$, $\omega_i(t)$ and $\omega_o(t)$ are all assumed to be bounded. In this
context, the following disturbance estimation observer can be constructed

\[
\begin{align*}
\ddot{\xi}(t) &= (\bar{A} - \bar{K}\bar{C})\xi(t) + \bar{B}u(t) - \bar{N}y(t) \\
\dot{\hat{x}}(t) &= \xi(t) + \ddot{S}^{-1}\bar{L}y(t)
\end{align*}
\]

(4.9)

where \(\xi(t) \in \mathbb{R}^{2n+r}\) is the state vector of the dynamic system above, \(\hat{x} \in \mathbb{R}^{2n+r}\) the estimate of \(\bar{x}\), \(\bar{S} = \bar{E} + \bar{L}\bar{C}\), and \(\bar{K} \in \mathbb{R}^{(2n+r) \times r}, \bar{L} \in \mathbb{R}^{(2n+r) \times r}\) are the gain matrices to be designed.

Here, we choose

\[
\bar{L} = \begin{bmatrix}
0_{n \times r} \\
0_{n \times r} \\
M_{r \times r}
\end{bmatrix}
\]

(4.10)

where \(M \in \mathbb{R}^{r \times r}\) is a non-singular matrix. One thus can calculate

\[
\bar{S} = \begin{bmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
MC & 0 & M
\end{bmatrix}, \quad \bar{S}^{-1} = \begin{bmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
-C & 0 & M^{-1}
\end{bmatrix}
\]

(4.11)

In terms of (4.7) and (4.11), it is further derived that

\[
\bar{A}\bar{S}^{-1}\bar{L} = I_n, \quad \bar{A}\bar{S}^{-1}\bar{L} = -\bar{N}.
\]

(4.12)

Using (4.12), the dynamic equation of the plant (4.8) can be expressed as

\[
\ddot{S}\ddot{x}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{G}\omega_i(t) + \bar{H}\dot{d}(t) + \bar{N}\omega_o(t) + \bar{L}\dot{y}(t)
\]

(4.13)

and the estimator (4.9) can be expressed as

\[
\ddot{S}\dot{x}(t) = \bar{A}\dot{x} + \bar{B}u(t) + \bar{K}(y(t) - \bar{C}\dot{x}(t)) + \bar{L}\dot{y}(t)
\]

(4.14)

or, alternatively,

\[
\ddot{S}\dot{x}(t) = (\bar{A} - \bar{K}\bar{C})\dot{x} + \bar{B}u(t) + \bar{K}y(t) + \bar{L}\dot{y}(t)
\]

(4.15)

Letting \(\bar{e}(t) = \bar{x} - \dot{x}(t)\) and subtracting (4.14) from (4.13) give

\[
\bar{e}(t) = \ddot{S}^{-1}[(\bar{A} - \bar{K}\bar{C})\bar{e}(t) + \bar{G}\omega_i(t) + \bar{H}\dot{d}(t) + \bar{N}\omega_o(t)]
\]

\[
= \ddot{S}^{-1}(\bar{A} - \bar{K}\bar{C})\bar{e}(t) + \bar{N}M^{-1}\omega_o(t) + \bar{H}\dot{d}(t) + \bar{G}\omega_i.
\]

(4.16)
From (4.16), one can choose a high-gain $M$ to reduce the effect from $\omega_o(t)$ on the estimation error dynamics.

According to the design algorithm proposed by [42], the high-gain matrix $\bar{K}$ can be computed as

$$\bar{K} = \bar{S}\bar{P}^{-1}\bar{C}^T,$$

(4.17)

where $\bar{P}$ is solved from the following Lyapunov equation

$$-(\mu I + \bar{S}^{-1}\bar{A})^T\bar{P} - \bar{P}(\mu I + \bar{S}^{-1}\bar{A}) = -\bar{C}^T\bar{C}$$

(4.18)

with $\mu > 0$ satisfying $\Re[\lambda_i(\bar{S}^{-1}\bar{A})] > -\mu, \forall i \in \{1, 2, ..., 2n + r\}$.

Now, with such a high-gain observer, the state $x$, the model disturbance $d$ and the output noise $\omega_o$ can be estimated simultaneously:

$$\hat{x}(t) = \begin{bmatrix} I_n & 0_n & 0_r \end{bmatrix} \hat{\bar{x}}(t)$$

(4.19)

$$\hat{d}(t) = \begin{bmatrix} 0_n & I_n & 0_r \end{bmatrix} \hat{\bar{x}}(t)$$

(4.20)

$$\hat{\omega}(t) = \begin{bmatrix} 0_n & 0_n & I_r \end{bmatrix} \hat{\bar{x}}(t)$$

(4.21)

### 4.3 High-gain observer based parameter estimation

In theory, if $(\hat{d}, \hat{x})$ is good enough, the identified $(\Delta A, \Delta B)$ will be adequately accurate. However, the disturbance estimation is not an easy task. According to our preliminary experiments, the delay between $x(t)$ and $\hat{x}(t)$ is very tiny and can be ignored. However, if the high-gain observer is not designed properly, the delay between $d(t)$ and $\hat{d}(t)$ may be obvious and can not be ignored.

For the sake of generality, it is useful to consider a time delay $\tau$ between the actual disturbance $d(t)$ (4.4) and its estimate $\hat{d}(t)$ (4.20). That is

$$\hat{d}(t + \tau) = d(t) + e_d(t)$$

(4.22)

where $e_d(t)$ is the estimation error, $\tau$ the time delay vector of $\hat{d}(t)$ with respect to $d(t)$. In this chapter and thereafter, ‘disturbance (estimation) delay’ is termed to the time delay $\tau$. If $\hat{d}(t)$ lags behind $d(t)$, then $\tau > 0$; if $\hat{d}(t)$ leads $d(t)$, then $\tau < 0$. Note that, since the high-gain observer is a causal system, $\tau > 0$.

Defining $\bar{e}(t) = \bar{x}(t) - \hat{\bar{x}}(t)$, the state estimation error $e_x(t) = x(t) - \hat{x}(t)$ can
be written as

\[ e_x(t) = [I_n \ 0_{n \times (n+1)}] \bar{e}(t). \] (4.23)

Recalling \( d(t) = \Delta A x(t) + \Delta B u(t) \) in (4.4), substituting \( x(t) = \hat{x}(t) + e_x(t) \) into (4.4) gives

\[ d(t) = \Delta A \hat{x}(t) + \Delta B u(t) + \Delta A e_x(t) \] (4.24)

The ARX model to be used for identifying \([\Delta A, \Delta B]\) is

\[ \hat{d}(t + \tau) = \Delta \hat{A} \hat{x}(t) + \Delta \hat{B} u(t) + e_{ARX}(t) \] (4.25)

where \( e_{ARX}(t) \) is the model prediction error, \([\Delta \hat{A}, \Delta \hat{B}]\) the estimate of \([\Delta A, \Delta B]\). Subtracting (4.25) from (4.24) gives

\[ e_d(t) = (\Delta A - \Delta \hat{A}) \hat{x}(t) + (\Delta B - \Delta \hat{B}) u(t) + \Delta A e_x(t) - e_{ARX}(t) \] (4.26)

As shown in [43], it can be proved that, if both \( e_d(t) \) and \( e_x(t) \) tends to desired small values as time approaches to infinity, minimising \( e_{ARX}(t) \) makes \([\Delta \hat{A}, \Delta \hat{B}]\) approach \([\Delta A, \Delta B]\). Furthermore, when model prediction error \( e_{ARX}(t) \) is minimised, \([\Delta \hat{A}, \Delta \hat{B}]\) turns into an optimal estimate of \([\Delta A, \Delta B]\).

From the viewpoint of system identification, the disturbance model (4.25) is in matrix form and can be rewritten as a set of multi-input-single-output submodels:

\[ \hat{d}_i(t + \tau) = \begin{bmatrix} \hat{x}^T(t) & u^T(t) \end{bmatrix} \theta_i, \quad i = \{1, 2, \ldots, n\} \] (4.27)

By rearranging the time delay \( \tau \), the submodel (4.27) is equivalent to

\[ \hat{d}_i(t) = \begin{bmatrix} \hat{x}^T(t - \tau) & u^T(t - \tau) \end{bmatrix} \theta_i, \quad i = \{1, 2, \ldots, n\} \] (4.28)

where

\[ \theta_i = \begin{bmatrix} \Delta \hat{A}_i^T \\ \Delta \hat{B}_i^T \end{bmatrix} \in \mathbb{R}^{n+p} \] (4.29)

\([\Delta \hat{A}_i, \Delta \hat{B}_i] \in \mathbb{R}^{1 \times (n+p)}\) represents the \(i\)th row of the matrix \([\Delta \hat{A}, \Delta \hat{B}] \in \mathbb{R}^{n \times (n+p)}\), and \(\{\theta_i\}\) is the parameter to be identified.

The submodel (4.28) is a special case of the first order \((n+p)\)-input-1-output ARX model with input delay \(\tau\) (note that the order of output \(\hat{d}_i(t)\) is zero). Thus, the estimation of \([\Delta A_i, \Delta B_i]\) turns into a system identification problem with time delay, and can be stated as follows: find the value \(\theta_i\) such that the following
objective function

\[
\min_{\theta_i} E(\theta_i) = \int_t e^2_{\text{ARX}}(t) dt = \int_t \left[ \hat{d}_i(t) - [\hat{x}^T(t - \tau) u^T(t - \tau)] \cdot \theta_i \right]^2 dt \quad (4.30)
\]

is minimised.

A lot of identification algorithms have been proposed for discrete ARX model. The continuous least squares estimation (LSE) is proposed for solving the ARX identification problem in the continuous domain \[137, 138, 43\]. For more information, please see \[139\].

Motivated by the work \[140\] for continuous-system parameter estimation, and with slight modifications, the \(i\)-th submodel \(128\) can be grouped into in to one \((n + p)\) order linear algebraic system:

\[
\begin{bmatrix}
\hat{x}^T(t - \tau) & u^T(t - \tau) \\
\hat{x}^T(t - \tau - \delta_1) & u^T(t - \tau - \delta_1) \\
\vdots & \vdots \\
\hat{x}^T(t - \tau - \delta_{n+p-1}) & u^T(t - \tau - \delta_{n+p-1})
\end{bmatrix}
\begin{bmatrix}
\theta_i(t) \\
\theta_i(t - \delta_1) \\
\vdots \\
\theta_i(t - \delta_{n+p-1})
\end{bmatrix}
= \begin{bmatrix}
\hat{d}_i(t) \\
\hat{d}_i(t - \delta_1) \\
\vdots \\
\hat{d}_i(t - \delta_{n+p-1})
\end{bmatrix}
\quad (4.31)
\]

where \(\delta_i = i\delta, i = 1, \ldots, n + p - 1\), \(\delta\) is some constant time interval, and \(t > (n + p)\delta + \tau\).

By using the well-known Cramer’s rule, the solution to \(131\) is

\[
\theta_i(t) = \frac{\text{Adj}(\Gamma_H)}{\det(\Gamma_H)}
\]

where \(\Gamma_H\) is defined in \(131\), \(\theta_i(t)\) is the \(j\)-th element of \(\theta_i(t)\), and \(\text{Adj}(\Gamma_H)\) is obtained by replacing the \(j\)-th column of \(\Gamma_H\) with the column vector \(\hat{d}_H\).

In order to avoid possible jumps at some points due to the numerical computation error, an integral is adopted

\[
\theta_i(t) = \frac{\int_{t-T_L}^t \det(\Gamma_H) \det(\Gamma_H) dt}{\int_{t-T_L}^t [\det(\Gamma_H)]^2 dt}, t > (n + p)\delta + T_L \quad (4.32)
\]

where \(t\) is the current time, and \(T_L\) is the selected integral length termed as integral window. In \(132\), the least-squares idea proposed by \[139\] is used actually.
In order to smooth the response curve further, we can take the mean calculation over some time interval $T_M$ as follows:

$$\bar{\theta}_i = \frac{1}{T_M} \int_{t-T_M}^{t} \theta_i(t) dt, \quad t \geq T_M$$  \hspace{1cm} (4.33)

where $\theta_i$ is identified from (4.32), the mean operation.

In this chapter, the parameter estimation without time-delay is the focus and it is assumed that $\tau = 0$. The issue of nonzero time delay and its compensation will be studied in next chapter.

An algorithm is proposed here to identify the parameter variation $[\Delta A, \Delta B]$ by using the high-gain observer technique. This algorithm can be stated as follows:

**Algorithm 4.1** (High-gain observer-based parameter estimation).

1) *Construct the augmented matrices as in (4.7) and select the observer gains $\bar{L}$ as in (4.10)*;

2) *Select $\mu$ as a quite large number (e.g., $\mu = 1000$) to make the disturbance delay $\tau$ small enough. Thus, it is safe to assume $\tau = 0$*;

3) *Solve the Lyapunov equation (4.18) and compute gain matrix $\bar{K}$ according to (4.17)*;

4) *Estimate $\hat{x}$, $\hat{d}$ in the forms of (4.19)-(4.20)*;

5) *Select time intervals $\delta$, $T_I$, $T_M$ and calculate equation (4.32)-(4.33) to get the parameters.*

### 4.4 Finite State Machine (FSM)-based adaptive change detection

The system parameters may change abnormally at a random time due to the degradation or the component failures, and detecting those changes plays an important role in condition monitoring. In the preceding section, the parameters of the plant have to keep unchanged during the identification interval $T_I + T_M$, such that the parameter estimation can converge to the actual value. Otherwise, the resulting parameter estimates may be wrong. In practice, however, it impossible
to guaranteed that $[\Delta A, \Delta B]$ does not change during every identification interval. Thus, identify the parameter as well as its change simultaneously is essential for improving the identification accuracy and reliability.

In this section, an adaptive change detection algorithm is proposed. This algorithm is implemented by the means of finite state machine (FSM). Like a rule based system, an FSM provides a simple and effective artificial intelligence technique to control a dynamic system, or describe the behaviors of an event-driven system, complex decision flows or supervisory logics. StateFlow® is a product of Mathworks, which extends Simulink® with a design environment for complex control and supervisory logic problems. With the aid of Stateflow®, the FSM of adaptive change detection will be designed and tested in this section.

### 4.4.1 Adaptive change detection

In this section, the simultaneous change detection and parameter estimation is termed as *adaptive parameter estimation*. The basic concept of adaptive parameter estimation is to detect the trend of parameter changes and update the corresponding thresholds. An adjustable threshold is adopted for detecting a sequence of changes. This technique overcomes the drawbacks of constant threshold and is really useful for real-time application.

In this study, it is assumed that $[\Delta A, \Delta B]$ may have multiple abrupt changes happening at any time. The idea of adjustable threshold is to detect the edge of each change and re-calculate the threshold thereafter. The adaptive threshold is expressed as the tolerable maximal relative varying ratio with respect to the *nominal value* of parameter. The nominal value can be derived from a healthy model created from the physical principles. In practice, the *nominal value* is usually determined by the mean value of the identified parameter under some selected steady period.

First, the nominal value over a pre-specified initial period is calculated to determine the initial threshold. If any parameter identified by Algorithm 5.3 keeps surpassing the threshold for a confirmed time interval, a change detection alarm is given. Then, a re-calculation of the new nominal value is triggered, and a new threshold is also given. The adaptive change detection and parameter estimation technique can be described formally by the following algorithm.

**Algorithm 4.2** (Adaptive parameter estimation).
1) Calculated the nominal value of the identified parameter as

\[
\theta_i^0 = \frac{1}{T_D} \int_{T_0}^{T_0 + T_D} \theta_i(t)dt
\]

where \(\theta_i(t)\) is given by (4.32), \(T_D\) the integral time and \(T_0\) is selected such that \(T_0 > (n + m)\delta + T_I + T_D\);

2) Set the adaptive thresholds for the changes detection:

\[
\theta^{+}_j = \begin{cases} 
(1 + \rho)\theta^0_j & \text{if } \theta^0_j > 0 \\
(1 - \rho)\theta^0_j & \text{if } \theta^0_j < 0 
\end{cases}
\]

\[
\theta^{-}_j = \begin{cases} 
(1 - \rho)\theta^0_j & \text{if } \theta^0_j > 0 \\
(1 + \rho)\theta^0_j & \text{if } \theta^0_j < 0 
\end{cases}
\]

where \(0 \leq \rho \leq 1\). \(\rho\) is the tolerable maximal varying rate with respect to the nominal value \(\theta^0_j\)

3) Select \(T_c\) as the so-called confirm window for the change. If the current time \(t\) satisfies

\[
\exists j, \tilde{\theta}_j(\tau) < \theta^{-}_j \text{ or } \tilde{\theta}_j(\tau) > \theta^{+}_j, \forall \tau \in \{\tau \mid t - T_c \leq \tau \leq t\}
\]

where the real-time parameter \(\tilde{\theta}_j(\tau)\) is given by (4.33), then a change is detected at time \(t\);

4) If any change is detected, calculate the following mean value:

\[
\hat{\theta}_j = \frac{1}{t - T_0} \int_{T_0}^{t} \theta_j(t)dt
\]

Set \(T_0 = t + T_N\), and go back to Step 1. Otherwise, continue Step 2 until the simulation ends.

**Remark 4.1.** The confirm window \(T_c\) in step 3 is used to avoid false alarm. It implies that, if and only if one of the parameter always surpasses the upper and/or lower thresholds during the time window, the change is confirmed. If \(\theta_j\) does not exceeds the thresholds at some time instant \(\tau\) (\(\tau\) is in the confirm window), then the change is ignored. \(T_c\) compromises between accuracy and response speed of change detection.
4.4.2 Finite State Machine implementation

The implementation of Algorithm 4.2 by using the Finite State Machine (FSM) technique and Stateflow is presented here. FSMs have been broadly applied to a lot of fields. It originated in the field of mathematics, and first emerged in the field of artificial intelligence for language representation. The FSM technique can be used both as a formal description of the solution and as a development tool for approaching and solving problems [141].

The FSM, also known as Finite State Automation (FSA), is a representation of an event-driven system for describing the system’s complex logics and behaviours. In such a system, the system transfers from one state (mode) to another prescribed state, provided that the condition defining the change is true. Generally, an FSM consist of 4 main elements:

1. **states** which define behavior and may produce actions;

2. **transitions and actions** which are movement from one state to another. During the transition process, some actions may be carried out;

3. **rules or conditions** which must be met to allow a state transition;

4. **events** which are either externally or internally generated, and may possibly trigger rules and lead to state transitions;

It is worthy noting the following facts of FSMs:

(1) A FSM must have an initial state which provides a starting point.

(2) A FSM must have one and only one active state (current state) which presents the current situation of a FSM. The input events are only received by the active state and processed according to the rules defined in the active state.

(3) Received input events act as triggers, which cause an evaluation of some kind of the rules that govern the transitions from the active state to other states. When a transition is finished, the destination state (at the end of the transition) becomes the active state, and the source state (which was active) turns into an inactive state.

(5) A FSM is one that has a limited or finite number of possible states. (An infinite state machine can be conceived but is not practical.)

The traditional way to represent relationships among the inputs, outputs, and states is to use truth tables. The resulting table describes the logic necessary
to control the behavior of the system under study. As shown in Figure 4.1, an alternative and better way to represent an FSM is to visualise it as a flow chart with conditional transitions among states. By using the flow chart, the concepts of states, conditions, actions, transitions can be easily expressed and the connections between states are more intuitive.

As shown in Figure 4.1, the state is represented by a square (e.g., the state **Initialisation**). The hollow arrow is the transition, which may associated with some conditions (e.g., \([t > T_0]\)) or events (e.g., \([\text{Under Threshold}]\)). These conditions and events are indicated by square brackets. When some event occurs and/or the condition is met, the active state of the FSM changes along the directed transition. An action is executed in two cases: (a) when the state transfers along the transition, the action associated with the transition, (e.g., \(\int_{T_0}^{T_0-T_D} \theta_j(t)dt\), is carried out; (b) if no transition takes place, the default action within the active state, (e.g., **Detect()**), is executed.

Figure 4.1 also represent the proposed adaptive change detection algorithm...
This FSM consists of five basic states (namely, Initialisation, Detection, Confirm, Trend Identification and GradualChg). There are four main actions: (a) \( \int_{T_0-T_D}^{T_0} \theta_j(t) dt \): to carry out Step 1 in Algorithm 4.2; (b) Set \( \theta_j^+ \), \( \theta_j^- \): to set the thresholds as Equations (4.35) and (4.36); (c) Detect(): to check whether \( \theta_j(\tau) \) exceeds the thresholds \( \theta_j^- \) and \( \theta_j^+ \); (d) IdentifyTrend(): to execute Step 4 and decide the trend of parameter variation. Two patterns of trend are defined in priori: TrnID=1, abrupt (step-like) change; TrndID=2, incipient (gradual) change.

Note that, (4.37) in Algorithm 4.2 is realised by combining the action Detect() and the transitions between states Detection and Confirm. This combination shows the advantage of FSM: an easy way to present complex logics and human knowledge.

Thus, Algorithm 4.2 is realised in the form of FSM. The hierarchical state ConstantChg in Figure 4.1 is the main body of Algorithm 4.2 for step-like change detection. An similar algorithm to detect gradual change is also realised by using FSM (see the state GradualChg at the bottom right). For the sake of insistence on on presenting theory development, the gradual change detection is skipped here.

With the fast development of computer techniques, a lot of softwares have been developed for realising the FSM. One of such softwares is Stateflow® using a variant of the finite state machine notation established by Harel [142]. Stateflow also integrates a C code generator enabling a fast development of real-time embedded system. With the aid of Stateflow, the complex logics in Algorithm 4.2 can be represented in a natural, readable, and understandable form. The whole FSM for adaptive change detection is shown in Figure 4.2.
Chp4. High-gain Observer based Parameter Estimation

Figure 4.2: Satellwəre based adaptive change detection
4.5 Case study: a gas turbine engine system

In this section, the proposed high-gain observer based parameter variation identification algorithm is applied to a gas turbine engine model. Both noise free case and noisy case are considered. The results are also compared to the standard subspace identification algorithm, namely \textit{n4sid}.

As identified in preceding chapter, the two shaft gas turbine engine model is given in the form of (4.3) with following coefficient matrices

\[
A_0 = \begin{bmatrix}
-0.9426 & -0.1601 \\
3.9439 & -3.2348
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
86.7941 \\
154.6907
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]  

(4.39)

where the input \(u(t)\) is the fuel flow rate, the output \(y(t)\) are the low-pressure shaft speed \(N_{lp}(t)\) and high-pressure shaft speed \(N_{hp}(t)\), respectively. Due to the degradation of the component, the actual GTE model may deviate from the nominal model \((A_0, B_0, C)\). The parameter variation \([\Delta A, \Delta B]\) is the parameters to be estimated. In this application, the following values are used to simulate the parameter variations:

\[
\Delta A = \begin{bmatrix}
\Delta a_{11} & \Delta a_{12} \\
\Delta a_{21} & \Delta a_{22}
\end{bmatrix} = \begin{cases}
\begin{bmatrix}
0.3 & 0.1 \\
0.8 & -0.8
\end{bmatrix} & t < 25, \\
\begin{bmatrix}
0.42 & 0.14 \\
1.12 & -1.12
\end{bmatrix} & 25 < t < 30, \\
\begin{bmatrix}
0.57 & 0.0 \\
1.52 & -1.52
\end{bmatrix} & 30 < t < 35,
\end{cases}
\]  

(4.40)

\[
\Delta B = \begin{bmatrix}
\Delta b_1 \\
\Delta b_2
\end{bmatrix} = \begin{cases}
\begin{bmatrix}
1.2 \\
-1.2
\end{bmatrix} & t < 20 \\
\begin{bmatrix}
1.44 \\
-1.44
\end{bmatrix} & 20 < t < 30, \\
\begin{bmatrix}
2.04 \\
-2.04
\end{bmatrix} & 30 < t < 35.
\end{cases}
\]  

(4.41)
These parameter variations are depicted in Figure 4.6 (a).

The input signal acquired from an engine test bed (sampling frequency: 40Hz) is used, as shown in Figure 4.3 (a). Note that, the input has been removed the mean value corresponding to the equilibrium of operating point. Its spectrum estimated by the 1024-point FFT (Fast Fourier Transform) is depicted in Figure 4.3 (b), where it can be seen that the energy of the input signal are mainly distributed over a certain frequency range [1, 2] Hz.

![Figure 4.3: The input signal to the gas turbine engine and its spectrum](image)

**High-gain observer design**

According to Algorithm 4.1, the augmented system in the form of (4.7) is first constructed. Then, by selecting

\[ L = \begin{bmatrix} 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}^T \]  

(4.42)

and \( \mu = 1000 \), the corresponding high gains \( \bar{K} \) is computed as

\[ \bar{K} = \begin{bmatrix} 0.011990144743945 & 0.000016747869854 \\ 0.000024299486689 & 0.011967323849854 \\ 7.995798132850000 & 0.015791835658789 \\ 0.000657288263037 & 7.986694758700000 \\ 0.000005997907031 & 0.00000004108208 \\ 0.00000004108206 & 0.000005993337500 \end{bmatrix} \times 10^{10} \]  

(4.43)

**A. Noise free case**

In this trial, it is assumed that the plant is noise free, that is, \( \omega_i = 0 \) and \( \omega_o = 0 \).
Figure 4.4 shows the state $x(t)$ and its estimate $\hat{x}(t)$. The disturbance $d(t)$ and its estimate $\hat{d}(t)$ are shown in Figure 4.5. It can be seen from these two figures that, the high-gain observer (4.43) gives quite good estimates of the states and disturbances. Some impulses can be seen clearly from the estimation errors (at $t=25$ and $30$ sec, respectively). This is caused by the parameter changes at those time instant (see equations (4.40)-(4.41) as well as Figure 4.6 (a)).

Figure 4.4: State estimation of the gas turbine engine (noise free).

Figure 4.5: The actual disturbance $d(t)$ and its estimate (noise free)

Figure 4.6 (a) shows the true parameter variations $[\Delta A, \Delta B]$ with respect to time, where one can see that the parameter variations have three abrupt changes at 20 sec, 25 sec, 30 sec, respectively. At 20 sec, only $\Delta B$ changes alone; at 30 sec, both $\Delta A$ and $\Delta B$ change simultaneously.

By using Algorithm 4.1 with $\delta = 0.2$ sec and $T_M = 3$ sec, the identified parameter variations $[\Delta \hat{A}, \Delta \hat{B}]$ are depicted in Figure 4.6 (b).
Figure 4.6: The actual parameter variation \([\Delta A, \Delta B]\) and its estimate \([\Delta \hat{A}, \Delta \hat{B}]\) (noise free).

The parameters of the adaptive change detection (Algorithm 4.2) are set as follows:

\[
T_D = 3 \text{ sec}, \rho = 20\%, \ T_c = 2 \text{ sec}; \quad (4.44)
\]

The adaptive change detection algorithm successfully finds the change and gives alarms at 22.014 sec, 27.011, 32.017 respectively. The delay of about two second is caused by the confirm window \(T_c = 2\).

For comparison, the results given by \textit{n4sid} (a subspace approach provided by Matlab) are listed in Table 4.1. Note that, the values shown in Table 4.1 are the mean values over corresponding intervals, rather than the real values at some time instant.

Comparing the identified values against the actual values in Table 4.1 one can see that these two methods have the similar performance in the noise free application. Particularly, in the noise-free case, \textit{n4sid} gives better estimates of \(\Delta A\) than the high-gain observer observer.

B. Subject to input/output noises

In this trial, let

\[
\omega_i = 0.2\sin(20t) + n_i(t) \quad (4.45)
\]

and

\[
\omega_o = \begin{cases} 
0, & t < 5 \\
0.05\sin(20t) + n_o(t), & t \geq 5 
\end{cases} \quad (4.46)
\]
Table 4.1: Comparison of different methods (noise free)

<table>
<thead>
<tr>
<th>Interval</th>
<th>Parameters</th>
<th>$\Delta a_{11}$</th>
<th>$\Delta a_{12}$</th>
<th>$\Delta b_1$</th>
<th>$\Delta a_{21}$</th>
<th>$\Delta a_{22}$</th>
<th>$\Delta b_2$</th>
</tr>
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<tr>
<td>5-20s</td>
<td>Actual</td>
<td>0.3</td>
<td>0.1</td>
<td>1.2</td>
<td>0.8</td>
<td>-0.8</td>
<td>-1.2</td>
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<td></td>
<td>n4sid</td>
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<td>0.1</td>
<td>1.1820</td>
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<td>-0.8</td>
<td>-1.093</td>
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<td>1.1206</td>
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<tr>
<td>20-25s</td>
<td>Actual</td>
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<td>1.44</td>
<td>0.8</td>
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<td>-1.44</td>
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<td></td>
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<td>0.1</td>
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<td>30-35s</td>
<td>Actual</td>
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<td>0.0</td>
<td>2.04</td>
<td>1.52</td>
<td>-1.52</td>
<td>-2.04</td>
</tr>
<tr>
<td></td>
<td>n4sid</td>
<td>0.57</td>
<td>0.0</td>
<td>2.0094</td>
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<td>-1.52</td>
<td>-1.8999</td>
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<tr>
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<td>HGO</td>
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<td>1.7930</td>
<td>1.4712</td>
<td>-1.4642</td>
<td>-1.7251</td>
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</table>

HGO: high-gain observer based algorithm

where $n_i(t)$ is a white noise with zero mean and the variance 0.0001; $n_o(t)$ a white noise with zero mean and the variance 0.00005. The estimates of plant states $x(t)$, disturbance $d(t)$ are shown in Figure 4.7 and 4.8, respectively. Because of the output noise added at 5 second and thereafter, it gives rises to the estimation errors. This phenomenon implies that the output noise has more impacts on the performance of the high-gain observer.

The estimated parameter variation $[\Delta \hat{A}, \Delta \hat{B}]$ are shown in Figure 4.9. The parameter variations identified by both n4sid and the proposed algorithm are listed in Table 4.2, where one can see that n4sid fails to give acceptable estimates of $[\Delta A, \Delta B]$. However, the proposed high-gain observer works well and has the similar performance compared to the noise free case. This ability to reject
4.6 Summary

In this chapter, by adopting the high-gain observer technique, the state and model disturbance are estimated simultaneously. With the ARX model formed by the disturbance estimate \( \hat{d}(t) \), state estimate \( \hat{x}(t) \) and input \( u(t) \), the time-varying parameter \([\Delta A, \Delta B]\) is then identified by using a continuous LSE method. Furthermore, an adaptive change detection approach is proposed and realised by means of FSM on the platform of Stateflow.
Due to the good model disturbance estimate achieved by the high-gain observer technique, the proposed algorithm is more robust to measurement noises compared to the standard subspace identification method $n4sid$. The proposed approach has been applied to a gas turbine engine, where real input data collected from test rig are used. The simulation results verifies these advantages.

In the high gain observer design, one critical thing to the performance of parameter estimation is the time delay between the actual variable and its estimate. It is worth to note that, in this chapter, the disturbance estimation time delay is assumed zero. From our preliminary simulation, however, the choice of $\mu$ shows a strong link with $\tau$. This assumption can be met by selecting a relative large $\mu$ (e.g., $\mu = 1000$) which makes $\tau$ small enough. In order to relax the constraints, the properties of the high-gain observer need further study. This issue will be discussed in next chapter giving some new sight on the high gain observer design.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Parameters</th>
<th>$\Delta a_{11}$</th>
<th>$\Delta a_{12}$</th>
<th>$\Delta b_1$</th>
<th>$\Delta a_{21}$</th>
<th>$\Delta a_{22}$</th>
<th>$\Delta b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-20s</td>
<td>Actual</td>
<td>0.3</td>
<td>0.1</td>
<td>1.2</td>
<td>0.8</td>
<td>-0.8</td>
<td>-1.2</td>
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<td></td>
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<tr>
<td>25-30s</td>
<td>Actual</td>
<td>0.42</td>
<td>0.14</td>
<td>1.44</td>
<td>1.12</td>
<td>-1.12</td>
<td>-1.44</td>
</tr>
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<td>1.0707</td>
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<td>30-35s</td>
<td>Actual</td>
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<td>0.0</td>
<td>2.04</td>
<td>1.52</td>
<td>-1.52</td>
<td>-2.04</td>
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<td>1.4644</td>
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<td>-1.7624</td>
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</tbody>
</table>

HGO: the high-gain observer based algorithm
Chapter 5

Parameter Estimation with Delay Compensation

In the preceding chapter, the time-varying parameters were identified by using the high-gain observer and adaptive change detection techniques, where the disturbance estimation delay $\tau$ is assumed to be zero. This assumption puts some constraints on the observer design, that is $\mu$ has to be selected carefully to make $\tau$ small enough. Hence, the time delay can be ignored and good parameter estimates can be reached. Unfortunately, there has been neither a criterion defining what $\tau$ is small enough, nor an explicit rule stating how $\mu$ should be selected. As a result, the observer have to be designed in a trial-and-error way involving more computation burden. In order to overcome these drawbacks and relax the constraints, this chapter takes into account a nonzero $\tau$.

In this chapter, we view the disturbance estimation as transfer functions from disturbance-to-estimate in the presence of a very general class of parameter variations as well as persistent excitation and measurement noise. A novel time delay calculation and compensation approach is proposed. The objective is to compute the disturbance estimation delay analytically and find a more accurate and robust solution than the TDE (Time Delay Estimation) approach.

This chapter is organised as follows: The properties of disturbance estimation in the high gain observer are examined in Section 5.2 followed by a constructive proof showing that the delay depends on the selection of the observer gain, but has nothing to do with the parameter variation. In Section 5.3 a novel algorithm is proposed to calculate the delay by means of Transfer Function Matrices (TFMs) and their phase responses. In order to compensate the nonlinear delay, in Section
5.4 A nonlinear phase delay (NLPD) filter approximation technique is proposed for delay compensation. As illustrated by a servo motor example in section 5.5, simulation results verify the calculation of the time delay and the improvement on parameter estimation. This delay computation and compensation algorithm, on one side, improves the performance of the high-gain observer based parameter estimation, and, on the other side, gives a new insight into the high-gain observer design.

5.1 Introduction

In theory, if the estimates $[\hat{d}, \hat{x}]$ obtained by the high-gain observer are good enough, the estimate of $[\Delta A, \Delta B]$ is adequately accurate. In the state estimation, the delay between the actual state and its estimate is very tiny and can be ignored without loss of estimation accuracy. In the disturbance estimation, however, if the high-gain observer is not designed properly, the delay may be obvious and can not be ignored. According to the preliminary experiments, because of the existence of delay $\tau$, the resulting parameter estimate is not acceptable.

Generally, if the input is a step signal, at the steady state no time delay exists between the actual disturbance signal and its estimate. However, according to the persistent excitation condition for system identification, using the step as a excitation signal may not give informative data for parameter estimation. Indeed, the input signal should cover the frequency range of interest. In this case, due to the nonzero disturbance estimation delay, the phase of the estimated signal $\hat{d}(t)$ differs from that of the actual signal $d(t)$. The different phases result in $\hat{d}(t)$ lags from $d(t)$. Even at the steady state, the delay does not disappear, which causes an incorrect parameter estimate.

One possible solution is to optimise the high-gain observer to make the time delay very small so that it can be ignored. As presented in Chapter 4, this was done by carefully selecting a good $\mu$ so that the time delay can be ignored. The main disadvantage of this approach is that the design process have to be a trial-and-error procedure and it is hard to guarantee a good performance.

Another possible solution is to admit the existence of delay, and estimate it by employing some time delay estimation (TDE) techniques provided by the system identification theory [68]. A lot of TDE methods have been proposed, e.g. cross correlation, pade approximation, explicit time-delay parameter methods,
area and moment methods, higher-order statistics methods. For more detailed information, see [98]. Although the TDE provides possibilities to estimate the time delay, a good delay estimation depends heavily on the amplitudes of the errors caused by the delay and that of parameters mismatch. Particularly, when the plant is subject to exogenous disturbances, it is impossible to distinguish the effects from the incorrect time delay and the errors from incorrect parameters. In some cases, the combination of a wrong estimate of time delay and wrong estimates of parameters may give a smaller prediction errors. It is worth to keep in mind that the best model approximation does not necessarily give the ”true” time delay [98]. Another drawback is that some TDE methods can only deal with the delay which is a multiple of the sampling interval. It is more difficult to estimate the time delay consisting of fractions of the sampling time.

An alternative is to calculate $\tau$ analytically by mathematical analysis and compensate it by phase-shifter. This approach can potentially offer the best of two worlds: adequate accuracy of delay estimation regardless of external disturbances, and relaxation on the observer design constraints and computation requirements. Fortunately, the analysis of TFM paves a way to study the properties of such a high-gain observer. In this chapter, it will be proved that the delay depends on the observer gain and $\mu$, but is immune from the parameter variation $[\Delta A, \Delta B]$. Thus, the time delay calculation is applicable for any size of model uncertainty, even in the case of severe model uncertainty and exoterical disturbances.

### 5.2 TFMs in disturbance estimation

In order to derive the mathematical expression of $\tau$, some TFMs relating high-gain observer and disturbance estimation is first formulated, and the relationship between $d(t)$ and its estimate $\hat{d}(t)$ is presented, followed by a proposition of a novel approach to determine $\tau$. Because the mathematical expression of $\tau$ is derived in our approach, the main advantage is that $\tau$ can be calculated accurately at the observer design stage and removed exactly by using phase compensation techniques.
5.2.1 TFM $G_{\hat{d}u}(s)$ relating $u(t)$ to $\hat{d}(t)$

From (4.15) and (4.20), one can write the estimation of $d(t)$ as

$$
\begin{align*}
\hat{\dot{x}}(t) &= (\hat{A} - \hat{K}\hat{C})\hat{x} + \hat{B}u(t) + \hat{K}y(t) + \hat{L}\dot{y}(t) \\
\hat{\dot{d}}(t) &= [0_{n\times n} I_{n\times n} 0_{r\times r}]\hat{x}
\end{align*}
$$

(5.1)

The Laplace transform gives the TFMs relating $u(t), y(t), \dot{y}(t)$ to $\hat{d}(t)$

$$
\hat{d}(s) = G_1(s)u(s) + G_2(s)y(s) + G_3(s)\dot{y}(s)
$$

(5.2)

where

$$
\begin{align*}
G_1(s) &= [0_{n\times n} I_{n\times n} 0_{r\times r}] \cdot (s\hat{S} - (\hat{A} - \hat{K}\hat{C}))^{-1} \cdot \hat{B} \\
G_2(s) &= [0_{n\times n} I_{n\times n} 0_{r\times r}] \cdot (s\hat{S} - (\hat{A} - \hat{K}\hat{C}))^{-1} \cdot \hat{K} \\
G_3(s) &= [0_{n\times n} I_{n\times n} 0_{r\times r}] \cdot (s\hat{S} - (\hat{A} - \hat{K}\hat{C}))^{-1} \cdot \hat{L}
\end{align*}
$$

(5.3) (5.4) (5.5)

(1) Considering the plant system subject to parameter variation $[\Delta A \Delta B]$ (4.2), the actual plant becomes

$$
\begin{align*}
\dot{x}(t) &= (A_0 + \Delta A)x(t) + (B_0 + \Delta B)u(t) + \omega_i(t) \\
y(t) &= Cx(t) + \omega_o(t)
\end{align*}
$$

(5.6)

and the TFM relating $u(t)$ to $y(t)$ is

$$
G_{yu}(s) = C[sI - (A_0 + \Delta A)^{-1}](B_0 + \Delta B)
$$

(5.7)

and

$$
y(s) = G_{yu}(s) \cdot u(s).
$$

(5.8)

(2) By differentiating the output equation of (5.6), the derivative of $y(t)$ is obtained as

$$
\dot{y}(t) = C\dot{x}(t) + \dot{\omega}_o
$$

(5.9)

Substituting the state equation of (5.6) into the derivation equation above gives

$$
\begin{align*}
\dot{x}(t) &= (A_0 + \Delta A)x(t) + (B_0 + \Delta B)u(t) + \omega_i(t) \\
\dot{y}(t) &= C[(A_0 + \Delta A)x + (B_0 + \Delta B)u] + \dot{\omega}_o
\end{align*}
$$

(5.10)
It follows that the TFM relating $u(t)$ to $\hat{y}(t)$ is

$$\hat{y}(s) = G_{\hat{yu}}(s) \cdot u(s) \quad (5.11)$$

where

$$G_{\hat{yu}}(s) = C(A_0 + \Delta A)[sI - (A_0 + \Delta A)]^{-1}(B_0 + \Delta B) + C(B_0 + \Delta B) \quad (5.12)$$

Substituting equations (5.8), (5.11) into (5.2) gives TFM $G_{\hat{du}}(s)$ relating $u(t)$ to $\hat{d}(t)$:

$$G_{\hat{du}}(s) = G_1(s) + G_2(s)G_{yu}(s) + G_3(s)G_{\hat{yu}}(s) \quad (5.13)$$

or, alternatively,

$$G_{\hat{du}}(s) = G_1(s) + G_2(s)C[sI - (A_0 + \Delta A)]^{-1}(B_0 + \Delta B)$$

$$+ G_3(s)\{C(A_0 + \Delta A)[sI - (A_0 + \Delta A)]^{-1}(B_0 + \Delta B) + C(B_0 + \Delta B)\}. \quad (5.14)$$

### 5.2.2 TFM $H_{du}(s)$ relating $u(t)$ to $d(t)$

Recalling the plant (5.6) with parameter variations and disturbance $d(t) = \Delta Ax(t) + \Delta Bu(t)$, the dynamics of $d(t)$ are governed by

$$\begin{align*}
\dot{x}(t) &= (A_0 + \Delta A)x(t) + (B_0 + \Delta B)u(t) + \omega_i(t) \\
\dot{d}(t) &= \Delta Ax(t) + \Delta Bu(t)
\end{align*} \quad (5.15)$$

Thus, the TFM relating $u(t)$ to $d(t)$ is

$$d(s) = H_{du}(s) \cdot u(s) \quad (5.16)$$

where

$$H_{du}(s) = \Delta A[sI - (A_0 + \Delta A)]^{-1}(B_0 + \Delta B) + \Delta B \quad (5.17)$$

Now both the TFMs relating $u(t)$ to $\hat{d}(t)$ and $d(t)$ have obtained. Next, the time delay between $\hat{d}$ and $d$ will be given by examining the relationship between $H_{du}(s)$ and $G_{\hat{du}}(s)$. 
5.3 Independence of time delay

For the sake of notation, some abbreviations are first defined as follows:

\[ \Lambda = (sI - A_0) \]  
\[ \Psi = [sI - (A_0 + \Delta A)] \]

\[ F(s) = G_2(s)CA^{-1} + G_3(s)[CA_0A^{-1} + C] \]

where \( F(s) \) is a \( n \times n \) TFM.

In the following, a constructive proof of \( G_{dd}(s) = F(s) \) will be presented, where \( G_{dd}(s) \) is the TFM relating \( d(t) \) to its estimate \( \hat{d}(t) \). Before proceeding, three lemmas are proved as follows.

**Lemma 5.1.** Given the plant \((A_0, B_0, C)\), if its corresponding high-gain observer \((4.9)\) is stable, then the following TFM

\[ G_1(s) + G_2(s)[CA^{-1}B_0] + G_3(s)[CA_0A^{-1}B_0 + CB_0] = 0_{n \times n} \]

**Proof.** This can be proved by analysing the dynamics of the high-gain observer. The plant system (with parameter variations) and its associated observer can be decomposed as shown in Figure 5.1, where the plant output \( y(t) \) is split into two parts: \( y_m(t) \) (from the nominal model \((A_0, B_0, C)\)), and \( y_u(t) \) (from the unmodelled dynamics). The same applies to \( d(t) \).

\[ y(t) = y_m(t) + y_u(t), \quad d(t) = d_m(t) + d_u(t) \]

where \( y(t) \) is measurable, and \( d(t) \) is unmeasurable. It is worthy noting that \( d_m(t) \) is always zero,

\[ d_m(t) \equiv 0, \]

because there is no model uncertainty within the nominal model \((A_0, B_0, C)\). All the effects from parameter variation and exoterical disturbances fall into \( d_u(t) \) and \( y_u(t) \).

According to the additionality of linear system, the outputs of the high-gain observer can be decomposed in a similar way:

\[ \hat{y}(t) = \hat{y}_m(t) + \hat{y}_u(t), \quad \hat{d}(t) = \hat{d}_m(t) + \hat{d}_u(t) \]
where $\hat{d}_m(t)$ is the estimate of $d_m(t)$, $\hat{d}_u(t)$ the estimate of $d_u(t)$.

Observe that the terms appearing on the left-hand side of (5.21) have a special meaning: $C\Lambda^{-1}B_0$ is the TFM relating $u(t)$ to $y_m(t)$ and $CA_0\Lambda^{-1}B_0 + CB_0$ is the TFM relating $u(t)$ to $\dot{y}_m(t)$. (This can be proved in a similar manner as (5.6)-(5.12), and replace $[\Delta A, \Delta B]$ with zero matrix, as there is no parameter variation in the nominal model). Compared to (5.2), these terms on the left-hand side of (5.21) are exactly the TFM relating $u(t)$ to $\hat{d}_m(t)$.

Furthermore, if the observer is stable, the disturbance estimate $\hat{d}(t)$ is asymptotically stable. With the decomposition as shown in Figure 5.1, it can be stated that $\hat{d}_m(t)$ approaches $d_m(t)$ asymptotically. Since $d_m(t) = 0$, $\hat{d}_m(t)$ of a stable observer is zero asymptotically. This means that, no matter what the values of $u(t)$ and $[\Delta A \Delta B]$ are, $\hat{d}_m(t)$ is always zero. This implies that $\hat{d}_m(t)$ has nothing to do with $u(t)$. Thus, the corresponding TFM relating $u(t)$ to $\hat{d}_m(t)$ is a zero matrix.

**Lemma 5.2.** If the high-gain observer (4.9) is stable, then

$$G_1(s) + G_2(s)[C\Psi^{-1}B_0] + G_3(s)[CA_0\Psi^{-1}B_0 + CB_0] + G_3(s)[C\Delta A\Psi^{-1}B_0] = F(s)\cdot \Delta A^{-1}\Psi^{-1}B_0$$

(5.25)
Proof. With the aid of lemma (5.1), $G_1(s)$ can be expressed as

$$G_1(s) = -G_2(s)[CA^{-1}B_0] - G_3(s)[CA_0A^{-1}B_0 + CB_0]$$ (5.26)

In order to eliminate the term $G_1(s)$ in (5.25), substituting (5.26) into the left-hand side of (5.25) gives

$$-G_2(s)[CA^{-1}B_0] - G_3(s)[CA_0A^{-1}B_0 + CB_0] + G_2(s)[C\Psi^{-1}B_0] + G_3(s)[CA_0\Psi^{-1}B_0 + CB_0] + G_3(s)[C\Delta A\Psi^{-1}B_0]$$ (5.27)

Thus, the left-hand side of (5.25) is equivalent to

$$G_2(s) \cdot C(\Psi^{-1} - \Lambda^{-1})B_0 + G_3(s) \cdot C[A_0\Psi^{-1} + \Delta A\Psi^{-1} - A_0\Lambda^{-1}]B_0$$ (5.28)

Observing that

$$(\Psi^{-1} - \Lambda^{-1}) = [\Lambda^{-1}(\Lambda - \Psi)\Psi^{-1}] = \Lambda^{-1}\Delta A\Psi^{-1},$$ (5.29)

and taking some algebraic manipulation, one has

$$A_0\Psi^{-1} + \Delta A\Psi^{-1} - A_0\Lambda^{-1} = (A_0\Lambda^{-1} + I)\Delta A\Psi^{-1}.$$ (5.30)

By substituting (5.30) into (5.28), the left-hand side of (5.25) becomes

$$G_2(s) \cdot CA^{-1}\Delta A\Psi^{-1}B_0 + G_3(s) \cdot C[(A_0\Lambda^{-1} + I)\Delta A\Psi^{-1}]B_0$$ (5.31)

The right-hand side of (5.25) is

$$F(s) \cdot \Delta A\Psi^{-1}B$$

$$\Leftrightarrow \{G_2(s)CA^{-1} + G_3(s)[CA_0A^{-1} + C]\} \cdot \Delta A\Psi^{-1}B$$

$$\Leftrightarrow G_2(s) \cdot C[\Lambda^{-1}\Delta A\Psi^{-1}]B_0 + G_3(s) \cdot C[(A_0\Lambda^{-1} + I)\Delta A\Psi^{-1}]B_0$$ (5.32)

Comparing (5.31) and (5.32) gives the result of Lemma (5.2). $\square$

**Lemma 5.3.** If the high-gain observer (4.9) is stable, then

$$G_2(s)C\Psi^{-1}\Delta B + G_3(s)C[A_0\Psi^{-1} + I]\Delta B + G_3(s)C\Delta A\Psi^{-1}\Delta B$$

$$= F(s) \cdot (\Delta A\Psi^{-1}\Delta B + \Delta B)$$ (5.33)
Proof. According to the definitions of $\Psi$ and $\Lambda$, we have

\[
\Delta A + \Psi = \Lambda \\
\Delta A \Psi^{-1} + I = \Lambda \Psi^{-1} \\
\Lambda^{-1}(\Delta A \Psi^{-1} + I) = \Psi^{-1}.
\]

Keeping the equations above in mind, and substituting the definition of $F(s)$ into (5.33), the right-hand side of (5.33) is equivalent to

\[
F(s) \cdot (\Delta A \Psi^{-1} \Delta B + \Delta B) \\
= F(s) \cdot \Lambda \Psi^{-1} \Delta B \\
= G_2(s) \cdot C \Psi^{-1} \Delta B + G_3(s) \cdot C [A_0 + \Lambda] \Psi^{-1} \Delta B
\] (5.34)

Compared to (5.33), the term of $G_2(s) \cdot C \Psi^{-1} \Delta B$ has been readily obtained in the left-hand side of (5.33). Replacing $\Lambda$ with $\Delta A + \Psi$, the second term of (5.34) can be expressed as

\[
G_3(s) C [A_0 \Psi^{-1} + I ] \Delta B + G_3(s) C \Delta A \Psi^{-1} \Delta B
\]
\[
\Leftrightarrow G_3(s) \cdot C [A_0 \Psi^{-1} + I + \Delta A \Psi^{-1}] \Delta B \\
\Leftrightarrow G_3(s) \cdot C [A_0 \Psi^{-1} + \Lambda \Psi^{-1}] \Delta B \\
\Leftrightarrow G_3(s) \cdot C [A_0 + \Lambda] \Psi^{-1} \Delta B
\] (5.35)

Therefore, the left-hand side of (5.33) is equivalent to

\[
G_2(s) \cdot C \Psi^{-1} \Delta B + G_3(s) \cdot C [A_0 + \Lambda] \Psi^{-1} \Delta B
\] (5.36)

By comparing (5.36) with (5.34), the result of Lemma 5.3 follows.

\[\square\]

Now the main theorem of the relationship between $d(t)$ and $\hat{d}(t)$ is given as follows.

**Theorem 5.1.** Given plant (5.6) and its corresponding high gain observer with $\bar{K}$, $\bar{L}$ (4.9), the relationship between TFM $G_{\hat{d}u}(s)$ and $H_{du}(s)$ is

\[
G_{\hat{d}u}(s) = F(s) \cdot H_{du}(s)
\] (5.37)
where $F(s)$, $G_{\hat{d}u}(s)$ and $H_{du}(s)$ are defined in (5.20), (5.13) and (5.17), respectively.

Proof. Substituting $\Psi = sI - (A_0 + \Delta A)$, $\Lambda = sI - A_0$ into $G_{\hat{d}u}(s)$ (5.13) gives

$$G_{\hat{d}u}(s) = G_1(s) + G_2(s)C\Psi^{-1}(B_0 + \Delta B) + G_3(s) \cdot [C(A_0 + \Delta A)\Psi^{-1}(B_0 + \Delta B) + C(B_0 + \Delta B)]$$

$$= G_1(s) + G_2(s)C\Psi^{-1}B_0 + G_2(s)C\Psi^{-1}\Delta B + G_3(s) \cdot C[A_0\Psi^{-1} + I]B_0 + G_3(s) \cdot C\Delta A\Psi^{-1}\Delta B$$

(5.38)

Notice that the terms underlined are identical to those on the left-hand side of (5.25) in Lemma 5.2. Furthermore, observe that the rest terms in (5.38) are the same as the left-hand side of (5.33) in Lemma 5.3. Thus, substituting (5.25), (5.33) into (5.38) gives

$$G_{\hat{d}u}(s) = F(s) \cdot \triangle A^{-1}\Psi^{-1}B_0 + F(s) \cdot (\Delta A\Psi^{-1}\Delta B + \Delta B)$$

(5.39)

Recalling the definition of $H_{du}(s)$ (5.17), one has

$$G_{\hat{d}u}(s) = F(s) \cdot H_{du}(s)$$

(5.40)

Thus, the result of Theorem 5.1 follows.

**Theorem 5.2.** Given a plant $(A_0, B_0, C)$ with parameter variation $[\Delta A, \Delta B]$ and the corresponding high-gain observer (4.9), for any value of $[\Delta A, \Delta B]$, the TFM $G_{\hat{d}d}(s)$ relating $d(t)$ to $\hat{d}(t)$ is equal to $F(s)$, that is,

$$G_{\hat{d}d}(s) = F(s).$$

(5.41)
Proof. According to the definition of transfer function matrix, we have

\[ \hat{d}(s) = G_{du}(s)u(s). \]  

(5.42)

Substituting the result of Theorem 5.1 into (5.42) gives

\[ \hat{d}(s) = F(s)H_{du}(s)u(s). \]  

(5.43)

Recall (5.17), one has

\[ d(s) = F(s)d(s). \]  

(5.44)

According to the definition of TFM, equation (5.44) states that \( F(s) \) is the TFM relating \( d(t) \) to \( \hat{d}(t) \), and the result of Theorem 5.2 follows.

Two useful propositions come straight from Theorem 5.1 and 5.2.

**Proposition 5.1.** The TFM \( G_{\hat{d}d}(s) \) is independent from model uncertainty \([\Delta A, \Delta B]\).

**Proposition 5.2.** The disturbance estimation delay \( \tau \) is invariant, no matter what the parameter variations \([\Delta A, \Delta B]\) are.

From the definition of \( F(s) = G_2(s)C(sI - A_0)^{-1} + G_3(s)(CA_0(sI - A_0)^{-1} + C) \), it can be found that \( F(s) \) is determined by \( G_2(s), G_3(s) \) and \( (A_0, B_0, C) \) only. As shown in (5.3)-(5.5), \( G_2(s), G_3(s) \) do not contain variables \( \Delta A, \Delta B \). Thus \( G_{\hat{d}d}(s) \) is independent from the parameter variations. In another words, the TFM relating \( d(t) \) to \( \hat{d}(t) \) is invariant for any value of \([\Delta A, \Delta B]\).

The TFM \( F(s) \) is a \( n \times n \) matrix and can be expressed in a matrix form:

\[
\begin{bmatrix}
\hat{d}_1 \\
\hat{d}_2 \\
\vdots \\
\hat{d}_n
\end{bmatrix} =
\begin{bmatrix}
F_{11}(s) & F_{12}(s) & \cdots & F_{1n}(s) \\
F_{21}(s) & F_{22}(s) & \cdots & F_{2n}(s) \\
\vdots & \vdots & \ddots & \vdots \\
F_{n1}(s) & F_{n2}(s) & \cdots & F_{nn}(s)
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n
\end{bmatrix}
\]  

(5.45)

Alternatively, \( F(s) \) can be expressed in terms of magnitude and phase response matrices

\[ F(s) = M(s)/\Phi(s) \]  

(5.46)

where \( M(s) \) is the magnitude response matrix and \( \Phi(s) \) the phase response matrix, respectively. Both are of size \( n \times n \). The amplitude of \( \hat{d}(t) \) is given by the
product of $M(s)$ and $d(s)$, while the phase angle of $\hat{d}(t)$ differs from that of $d(t)$ by the amount of $Φ(s)$.

Consider an ideal disturbance observer where $\hat{d}_i(t)$ should be determined by $d_i(t)$ alone and has no coupling with $d_j(t), j \neq i$. That is, ideally, $M(s)$ is an identity matrix and $Φ(s)$ is a zero matrix.

In practice, $M(s)$ given by the high-gain observer is an identity matrix approximately, that is, $M_{ij}(s) \to 0, j \neq i$. Therefore, $\hat{d}_i(t)$ is dominantly determined by $d_i(t)$ alone. Hence, only the diagonal elements of $Φ(s)$ is considered in calculating the time delay $τ$.

Substituting $s = jω$ into $Φ(s)$, the delay function $τ_i(ω)$ with respect to frequency $ω$ is given by

$$τ_i(ω) = -\frac{Φ_{ii}(jω)}{ω}$$

(5.47)

where $τ_i(ω)$ is the time delay in seconds between $\hat{d}_i(t)$ and $d_i(t)$ at frequency $ω$. Note that, $τ_i(ω)$ is the delay on $i$-th disturbance estimation and $τ_i(ω)$ may be different from each other.

**Remark 5.1.** As $F(s)$ is independent from $[ΔA, ΔB]$, both $M(s)$ and $Φ(s)$ are also independent from the parameter variation. Hence, the time delay $τ_i(ω)$ does not change with respect to $[ΔA, ΔB]$. Then, it is practical to compute the delay by assigning an value to $[ΔA, ΔB]$ arbitrarily, and the calculated delay is applicable for any value of $[ΔA, ΔB]$.

**Remark 5.2.** $F(s)$ is readily obtained after the observer is designed and $\bar{K}$, $\bar{L}$ are determined. One can compute the time delay $τ_i(ω)$ accurately from the phase response matrix $Φ(s)$. Furthermore, a phase-shifter with $Φ(s)$ phase response can be properly designed to compensate the delay.

**Remark 5.3.** Generally, $τ$ is a function of frequency $ω$ and may vary at different frequency. If frequency components of $d(t)$ have different delay time at different frequency, phase-frequency distortion occurs which leads to a poor estimate. In order to avoid the phase-frequency distortion, the observer should be designed in a way such that the time-delay over the frequency range of interest is constant or varies slightly. In terms of phase response, it is required that $Φ(jω)$ is of linear phase, (that is, $Φ(jω)$ is a linear function of frequency).

Based on the analysis and discussion above, a time delay calculation algorithm is given as follows:
Algorithm 5.1 (Disturbance estimation delay calculation algorithm for high-gain observer).

1) Set the initial values of $\mu$ and $\alpha$, design a high gain observer (4.15), (4.19) and (4.20);

2) Compute the TFMs $G_2(s)$, $G_3(s)$ and $F(s)$ from (5.4), (5.5), (5.20), respectively;

3) Express $F(s)$ in terms of magnitude and phase response (5.46).

4) Calculate the delay $\tau_i(\omega)$, $i = 1, 2, \ldots, n$ as (5.47) over the frequency range $\Omega$, where $\Omega = [\omega_1, \omega_2]$ is the frequency range of interest. Generally, $\Omega$ is known a priori by input signal design or estimated by some frequency analysis technique;

5.4 Delay compensation by filter approximation

In the context of parameter identification, the purpose of delay compensation is to align the data pairs $([\hat{x}^T(t) \ u^T(t)], \hat{d}_i^T(t - \tau_i))$. The fundamental of the presented delay compensation procedure is to design a set of filters whose phase responses are equivalent to the phase responses of the disturbance estimation $\{\Phi_{ii}(s)\}$ respectively, then use these filters to delay $[\hat{x}^T(t) \ u^T(t)]$ with the same time $\tau_i$. Thus, the data pairs are aligned as $([\hat{x}^T(t - \tau) \ u^T(t - \tau)], \hat{d}_i^T(t - \tau))$. This scheme is shown in Figure 5.2 where $L(s)$ is the filter to be design to compensate the delay.

![Figure 5.2: The scheme of delay compensation and parameter estimation](image)

The key step in delay compensation is to choose the coefficients of the compensation filter $L(s)$ properly so that the desired phase-delay frequency response can
be achieved. In the following, the time delays related to the phase response are defined first, followed by a structure and parameter selection of the compensation filter.

5.4.1 Phase delay

Generally, a linear system changes the complex amplitude of the input signal, but the frequency is not changed. In terms of the magnitude-phase response, the estimation of $i$-th disturbance $d_i$ can be expressed as

\[
\begin{align*}
|\hat{d}_i(j\omega)| &= M_{ii}(j\omega) \cdot |d_i(j\omega)| \\
\angle\hat{d}_i(j\omega) &= \angle d_i(j\omega) + \Phi_{ii}(j\omega)
\end{align*}
\]

(5.48)

where $M_{ii}$ and $\Phi_{ii}$ is the $i$-th diagonal element of the magnitude response matrix $M$ (5.46) and the phase response matrix $\Phi$ (5.46), respectively. $| \cdot |$ denotes the magnitude, and $\angle$ the phase angle.

In (5.48), $\Phi_{ii}(j\omega)$ is typically referred to as the phase shift of the observer on $i$-th disturbance estimation. Due to the phase shift, the observer can change the relative phase relationships among the components of the disturbance estimate. This possibly results in significant modifications to the time domain characteristics of the estimate even when the gain $M$ is an identical matrix.

In our high-gain observer, $M$ is approximately an identity matrix giving an unit gain to the disturbance estimate. Therefore, no distortions are introduced to the magnitude estimation. However, the non-zero phase response makes the phase of $\hat{d}_i(t)$ differ from $d_i(t)$. This phase difference changes the disturbance estimate in an unwanted manner. Therefore, the undesirable effects in phase response (5.48) is termed as phase distortions.

Generally, a non-zero phase response leads to a non-zero time delay. From the viewpoint of signal processing, phase response in the frequency domain can be interpreted as time delay in the time domain. There are two kinds of time delay associated with phase response: a) **Phase Delay**; b) **Group Delay**. As group delay is for the modulated signal $g(t) = y(t)\cos(\omega t)$ and we concern the time delay between $d(t)$ and $\hat{d}(t)$ (neither of them is modulated signal), it is the phase delay that will be examined in this chapter.

**Definition 5.1. Phase Delay** The phase delay of a filter $F_i(s)$ with phase
response $\Phi_i(j\omega)$ is defined by

$$\tau_i \triangleq -\frac{\Phi_i(j\omega)}{\omega} \quad (5.49)$$

The phase delay is also referred to as pure delay or transport lag. In our study, time delay is also used to refer phase delay. The phase delay gives the time delay in seconds.

For a linear phase response (which means the phase shift is a linear function of $\omega$), the phase delay is a constant over all the frequencies. If the phase response is a nonlinear function of $\omega$, then the components of $d(t)$ at different frequencies will be shifted in a manner that results in changing their relative phases. When these exponentials are superimposed, it estimate $\hat{d}(t)$ will look considerably different from the input signal.

### 5.4.2 Magnitude-phase approximation

As discussed before, one possible way to align the data pairs is to lag the signals $[\hat{x}^T(t) \ u^T(t)]$ by $\tau(\omega)$, where $\tau(\omega)$ is the phase delay (time delay) introduced by the observer.

Usually, phase delay $\tau$ is a nonlinear function of $\omega$ (see Figure 5.6). This is the so-called nonlinear phase delay (NLPD) and it is the source of phase distortions. NLPD means the observer lags/leads the components of disturbance $d(t)$ at different frequencies with different delay time. Thus, the delay compensation here is not a linear phase shifting problem (which lags the signal with the same delay at all frequencies). Indeed, the compensator to be designed should approximate the nonlinear phase response of the observer, but do not change the magnitude of $[\hat{x}^T(t) \ u^T(t)]$.

It is intuitive to compensate the delay $\tau(\omega)$ by designing an all-pass filter, whose magnitude response is unit and phase response is the same as $F(s)$. Some examples are, transfer function approximation approach in analog filters design \[143\] or digital filters design \[144\].

Although the all-pass filter has been successfully applied to compensate the group delay, it is not suitable for phase delay compensation. Particularly, it is true for a short phase delay. This point will be proved in the following subsection.
Possibility of all-pass filter implementation

An all-pass filter $A(s)$ is a causal, stable system with unit magnitude response:

$$|A(s)| = 1$$

(5.50)

One important property of the all-pass filter is that its zeros and poles are symmetric with respect to the imaginary axis $j\omega$. Hence, the transfer function of the all-pass filter has a general form

$$A(s) = \frac{\gamma(-s)}{\gamma(s)}$$

(5.51)

where $\gamma(s)$ is a Hurwitz polynomial. It is then given that the phase response $\Phi_A(\omega)$ of an all-pass filter $A(s)$ is equal to twice the phase response of its numerator. That is

$$\Phi_A(\omega) = \angle\gamma(-j\omega) - \angle\gamma(j\omega)$$

(5.52)

For a real stable filter, poles have to be on the left-half $s$-plane and complex poles must occur in conjugate pairs. Since the poles and zeros of an all-pass filter are symmetric with respect to the imaginary axis $j\omega$, with some algebraic and triangle manipulation, one has $\angle\gamma(-j\omega) = n\pi - \angle\gamma(j\omega)$, where $n$ is the number of pole-zero pairs. The phase response $\Phi_A(\omega)$ of the all-pass filter is

$$\Phi_A(\omega) = n\pi - 2\angle\gamma(j\omega)$$

(5.53)

And the corresponding phase delay is

$$\tau_p(\omega) = -\frac{n\pi}{\omega} + \frac{2\angle\gamma(j\omega)}{\omega}$$

(5.54)

From the expression (5.53), it can be seen that there is a constant phase shift $n\pi$ in the phase response of all-pass filter. As to the high-gain observer, the phase delay to be compensated is relative small (in order of 0.1 sec over the frequency range of interest). Because the angle $\angle\gamma(j\omega)$ in (5.54) is in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is impossible (or at least, very very hard) to cancel the negative value $-n\pi/\omega$ by selecting $\gamma(j\omega)$.

Now, a conclusion can be drawn: the all-pass filter for small phase delay compensation is unobtainable. The nonlinear-phase delay approximation technique...
has to be employed to compensation the nonlinear disturbance estimation delay.

**Nonlinear-phase delay approximation**

Since the TFM $F(s)$ relating $\hat{d}(t)$ to $d(t)$ has been calculated, a straight way to compensate the delay is to reuse $F(s)$ as the compensator to lag $[\hat{\hat{x}}(t) u(t)]$. However, the order of $F(s)$ given by Algorithm (5.1) may be very high and the realisation of $F(s)$ will occupy too many computational resources. It is necessary to approximate $F(s)$ with a reduced order system. Therefore, the problem of delay compensation turns into an filter approximation problem. That is, given desired frequency response $F(s)$ over a frequency domain $\Omega$, to find an optimal filter with the similar magnitude response, phase response and phase delay as $F(s)$.

Generally, $\Omega$ is termed as *design frequency domain* consisting of disjoint sets $\Omega^P$ (the *passband*), $\Omega^T$ (the *transition band*) and $\Omega^S$ (the *stopband*). $F(s)$ defines a desired magnitude response $|F(j\omega)|$, a desired phase response $\angle(F(j\omega))$ and a desired phase delay response $-\angle(F(j\omega))$, where the last two functions are defined on $\Omega^P$ only.

An $n$-th order filter $L(s)$, which is to be designed for approximating $F(s)$, can be expressed as

$$ L(s) = K \frac{b(s)}{a(s)} = K \frac{s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \quad (5.55) $$

where $\{a_i\}$ and $\{b_i\}, i = 0, 2, \ldots n - 1$ are real coefficients for synthesising a realisable filter, and $a(s)$ is a Hurwitz polynomial.

With some prescribed norms (typically, the least-squares $l_2$-norm or the maximum norm $H_\infty$-norm), the delay compensation can be stated as Magnitude and NonLinear Phase Delay (M-NLPD) approximation.

Let $x$ denote the vector of parameters $\{a_i\}$ and $\{b_i\}, i = 0, 2, \ldots n - 1$, that is

$$ x = [a_0, a_1, ..., a_{n-1}, b_0, ..., b_{n-1}], $$

the (M-NLPD) approximation problem to be solved can be formulated as the minimisation of magnitude error

$$ \min_{x \in \mathbb{R}^{2n}} \| V_1(\omega) \cdot (M(j\omega) - |L(j\omega, x)|) \|_{\Omega, p}, \quad (5.56) $$
phase error
\[
\min_{x \in \mathbb{R}^{2n}} \| V_2(\omega) \cdot [\Phi(j\omega) - \angle(L(j\omega, x))] \|_{\Omega_P, p}, \tag{5.57}
\]
and phase delay error
\[
\min_{x \in \mathbb{R}^{2n}} \| V_3(\omega) \cdot [-\frac{\Phi(j\omega)}{\omega} + \frac{\angle(L(j\omega, x))}{\omega}] \|_{\Omega_P, p} \tag{5.58}
\]
where \( M(j\omega) \) is the desired magnitude response of \( F(s) \), \( \Phi(j\omega) \) the desired phase response of \( F(s) \). \( V_1(\omega), V_2(\omega) \) and \( V_3(\omega) \) are spectrum-weighting functions. The operator \( \| \cdot \|_{\Omega, p} \) denotes the operation of the \( l^p \)-norm over frequency range \( \Omega \):
\[
\| f \|_{\Omega, p} := \begin{cases} 
\left\{ \sum_{\omega \in \Omega} |f(\omega)|^p \right\}^{1/p} & \text{if } 1 \leq p < \infty \\
\max_{\omega \in \Omega} |f(\omega)| & \text{if } p = \infty
\end{cases}
\]

\( V_1(\omega), V_2(\omega) \) and \( V_3(\omega) \) are determined by the spectrum analysis of the plant input signal. The spectrum of the input signal \( u(t) \) can be defined as
\[
P_u(\omega) = \int_{-\infty}^{\infty} R_{uu}(\tau)e^{-j\tau\omega}d\tau \tag{5.59}
\]
where \( R_{uu}(\tau) \) is the (continuous) autocorrelation function of \( u(t) \):
\[
R_{uu}(\tau) = \int_{-\infty}^{\infty} u(t)u(t - \tau)dt \tag{5.60}
\]

Thus, the weighting function \( V_i(\omega), i = 1, 2, 3 \) can be calculated as
\[
V_i(\omega) = v_i \cdot P_u(\omega) \tag{5.61}
\]
where \( v_i, i = 1, 2, 3 \) are scalars reflecting the trade-off between the magnitude error (5.56), phase error (5.57) and phase delay error (5.57). Thus, the objective function becomes
\[
\| V_1(\omega) \cdot (M(j\omega) - |L(j\omega, x)|) \|_{\Omega_P} + \| V_2(\omega) \cdot [\Phi(j\omega) - \angle(L(j\omega, x))] \|_{\Omega_P, p} \\
+ \| V_3(\omega) \cdot [-\frac{\Phi(j\omega)}{\omega} + \frac{\angle(L(j\omega, x))}{\omega}] \|_{\Omega_P, p} \tag{5.62}
\]

Furthermore, the passband \( \Omega^p \) can be determined according to the spectrum of input signal. Assume the frequency band of input signal is limited within
\[ \Omega^P = [\omega_1, \omega_2] \]

In order to simplify the expression of the objective function, let \( \Phi(j\omega) = \angle F(j\omega) \) on \( \Omega^P \) be the desired phase response, we have

\[
F(j\omega) = |F(j\omega)| e^{j\Phi(j\omega)}, \quad \omega \in \Omega^P
\]

Recalling the phase error (5.57) on \( \Omega^P \), it is convenient to rotate \( F(j\omega) \) and \( L(j\omega) \) through the angle \(-\Phi(j\omega)\) and to use instead

\[
\bar{F}(j\omega) = e^{-j\Phi(j\omega)} |F(j\omega)| e^{j\Phi(j\omega)}
\]

\[
\bar{L}(j\omega) = e^{-j\Phi(j\omega)} L(j\omega, x)
\]

For \( \omega \in \Omega^P \), one then has

\[
|\bar{F}(j\omega)| = |F(j\omega)|, \quad \angle \bar{F}(j\omega) = 0 \quad \text{and} \quad |\bar{L}(j\omega, x)| = |L(j\omega, x)|, \quad \Phi(j\omega) - \angle (L(j\omega, x)) = -\angle \bar{L}(j\omega, x) \tag{5.63}
\]

It can be seen that \( F(j\omega) \) has the same magnitude as \( F(j\omega) \) over \( \Omega^P \). So does \( \bar{L}(j\omega, x) \) and \( L(j\omega, x) \). Furthermore, if \( L(j\omega, x) \neq 0 \) over \( \Omega^P \), from (5.63), one gets that the phase error functions (5.57) is equivalent to

\[
\min_{x \in \mathbb{R}^n} ||V_2(\omega) \cdot \angle \bar{L}(j\omega, x)||_{\Omega^P, p} \tag{5.64}
\]

Substituting (5.63) into the phase delay error function (5.58) gives

\[
\min_{x \in \mathbb{R}^n} ||V_3(\omega) \cdot \frac{\angle \bar{L}(j\omega, x)}{\omega}||_{\Omega^P, p} \tag{5.65}
\]

Thus, the objective function (5.62) can be rewritten as

\[
\min_{x \in \mathbb{R}^n} ||V_1(\omega) \cdot (M(j\omega) - |L(j\omega, x)|)||_{\Omega, p} + ||V_2(\omega) \cdot \angle \bar{L}(j\omega, x)||_{\Omega^P, p} + ||V_3(\omega) \cdot \frac{\angle \bar{L}(j\omega, x)}{\omega}||_{\Omega^P, p} \tag{5.66}
\]

Now the proposed magnitude and nonlinear phase delay (M-NLPD) compensation filter design algorithm can be given as follows:
Algorithm 5.2 (Magnitude and Nonlinear Phase Delay (M-NLPD) Compensation Filter Design).

1) Compute the desired filter \( F(s) \) by using Algorithm 5.1 and express \( F(s) \) in terms of magnitude and phase response (5.46).

2) Compute \( P_u(\omega) \), as (5.59), the spectrum of the input signal \( u(t) \).

3) Determine the desired pass band \( \Omega^P = [\omega_1, \omega_2] \) according to the spectrum \( P_u(\omega) \). The pass band should cover the frequencies where most of the signal energy is. (e.g., 99% of the signal energy).

4) Specify the stop band \( \Omega^S \), transition band \( \Omega^T \) and the magnitude requirements over these two bands.

5) Select the values of \( v_1, v_2 \) and \( v_3 \) according to the trade-off between the magnitude error and phase delay error, and determine the weighting function \( V_1(\omega), V_2(\omega) \) and \( V_3(\omega) \) by (5.59), (5.60) and (5.61).

6) For each \( F_i(s), i = 1, \ldots, n \), (which is the diagonal element of TFM \( F(s) \)), design a filter \( L_i(s) \), (5.55 by selecting the filter order and optimising the objective function; (5.66).

Remark 5.4. Step 4 can be carried out according to experiences or the severity degree of noise corruption as well as the realisation possibility. It is better to improve the noise attenuation performance by specifying a narrower transition band and lower magnitude over stop band, but it faces the increase of filter complexity (filter order) and computation costs.

5.4.3 Parameter estimation with delay compensation

By taking the disturbance estimation delay \( \tau \) into account, the parameter estimation algorithm 4.1 now is extended to the case of nonzero \( \tau \):

Algorithm 5.3 (High-gain observer-based parameter estimation with delay compensation).

1) Set the initial values of \( \mu \), design a high gain observer (4.15) by solving the Lyapunov equation (4.18) and computing the high-gain matrix \( \bar{K} \) (4.17), \( \bar{L} \) (4.10);

2) Apply Algorithm 5.1 to estimate the time delay \( \tau_i(\omega), i = 1, \ldots, n \);
3) Apply Algorithm 5.2 to design a bank of compensators \( \{ L_i(s) \} \) (5.55), \( i = 1, \ldots, n \), which approximates the disturbance delay \( \tau_i(\omega) \), respectively;

4) Connect \( L_i(s) \) to \( \hat{x}(t) \) and \( u(t) \) to shift them by \( \tau_i(\omega) \) for delay compensation, as shown in Figure 5.2;

5) Implement the real-time estimation using the constructed observer (4.9) and compensation filter \( L_i(s) \), and obtain \( u(t - \tau_i), \hat{x}(t - \tau_i) \) and \( \hat{d}_i(t) \) in the forms of (4.28);

6) Apply Algorithm 4.1 to identify the parameters.

From Figure 5.2 it can be seen that the major difference between Algorithm 5.3 and 4.1 is the employment of compensation filter \( L(s) \) to lag \( u(t) \) and \( \hat{x}(t) \). In this way, the data pair \([\hat{x}(t), u(t), \hat{d}(t)]\) is aligned and the correct estimate of parameter variation \([\Delta A, \Delta B]\) is obtainable.

### 5.5 Case Study: a servo motor system

In this section, the proposed delay calculation and compensation approach is applied to a three-mass servo system and a parison between Algorithm 4.1 and Algorithm 5.3 is given.

A reduced order model of a three-mass servo system can be expressed by (5.6) with the following coefficient matrices

\[
A_0 = \begin{bmatrix} -1.0830 & -0.0453 \\ 0.1004 & 0.0014 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.9540 \\ 0.0140 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \tag{5.67}
\]

where the input \( u(t) \) is the supply voltage to the servo motor, the output \( y(t) \) a two elements vector of the shaft speed and load speed, respectively. The input and its spectrum are shown in Figure 5.3 where it can be seen that \( u(t) \) consists of a sum of sinusoids over certain band \( \Omega = \{0.02, 0.08, 0.146, 0.205, 0.273, 0.332, 0.4, 0.459, 0.527, 0.586, 0.654, 0.713, 0.781, 0.840, 0.908, 0.967\} \text{ (Hz)} \).

These coefficient matrices (5.67) are accurate when the motor is in good condition. In the meanwhile, due to the usage and degradation, \( \Delta A \) and \( \Delta B \) are non-zero matrices. These matrices are the parameters to be estimated. In this
simulation, they are assumed

\[
\Delta A = \begin{bmatrix}
-0.1083 & -0.0045 \\
0.0100 & 0.0001 \\
\end{bmatrix}, \quad \Delta B = \begin{bmatrix}
-0.0954 \\
-0.0014 \\
\end{bmatrix}.
\] (5.68)

**High gain observer design**

A high gain observer is first constructed with arbitrarily selected \( \mu \), then the disturbance delay is calculated and a filter is designed to compensate the delay.

Here, we choose \( \alpha = 10 \) and \( \mu = 25 \), and a high gain observer is constructed according to Algorithm 4.1. Note that, in previous section, \( \mu \) was required to be at least 1000 so that the disturbance delay became small enough and a good identification result can be obtained. The resulting high gains \( \bar{K} \) corresponding to \( \mu = 25 \) is

\[
\bar{K} = \begin{bmatrix}
0.072218014418967 & 0.000028559909589 \\
0.000099829881472 & 0.074903803193174 \\
1.220475974788427 & 0.002424858528359 \\
-0.001211932277104 & 1.247539778727532 \\
0.00147633899510 & 0.000000518459801 \\
0.000000518459801 & 0.001498030341864 \\
\end{bmatrix} \times 10^6
\] (5.69)
Time delay analysis

By using the proposed Algorithm 5.1, the transfer function matrix $F(s)$ (5.20) are computed:

$$F(s) = \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix}$$

(5.70)

where

$$F_{11}(s) = \frac{122048(s + 49.9)(s + 1.079)(s + 0.002775)(s^2 + 100s + 2500)}{\text{Den}(s)}$$

$$F_{12}(s) = \frac{242.49(s + 50)(s + 49.9)(s + 1.079)(s + 0.002776)(s - 8.101e - 007)}{\text{Den}(s)}$$

$$F_{21}(s) = \frac{-121.19(s + 50)(s + 49.9)(s + 1.079)(s + 0.002768)(s + 7.015e - 006)}{\text{Den}(s)}$$

$$F_{22}(s) = \frac{124754(s + 50)(s + 49.9)(s + 48.92)(s + 1.079)(s + 0.002775)}{\text{Den}(s)}$$

and the common denominator is

$$\text{Den}(s) = (s+50.1)(s+48.92)(s+1.079)(s+0.002775)(s^2+99.67s+2484)(s^2+100s+2501).$$

The bode plots (magnitude response and phase response) of $F(s)$ (5.70) are depicted in Figure 5.4 and Figure 5.5 respectively.

From Figure 5.4 it can been seen that the magnitude response $M_{ii}(\omega), i = 1, 2$ is 0 dB over [0,10] (rad/sec). Contrarily, the magnitude $M_{ij}(\omega), i \neq j$ is lower than -50 dB which is very very small compared to that of $M_{ii}(\omega)$. Hence, it verifies that $M(\omega)$ can be treated as an identity matrix. Therefore, $\hat{d}_i(t)$ is dominated by $d_i(t)$ and the time delay is determined by $\Phi_{ii}(\omega)$. 
Figure 5.4: Magnitude response of $F(s)$

Figure 5.5: Phase response of $F(s)$
According to the spectrum analysis of the input (see Figure 5.3), the frequency range of $d(t)$ is $\Omega = [0, 1]$ Hz. The resulting time delay $\tau(\omega)$ (5.47) over the frequency range $\Omega$ is shown in Figure 5.6. It can be found that, the time delay decreases in a non-linear fashion as the frequency increases. Over the frequency range $[0, 1]$ (Hz), both $\tau_1$ and $\tau_2$ are around 0.06 second and the variation is not very large.

The disturbance $d(t)$ and its estimate $\hat{d}(t)$ are shown in Figure 5.7 where a time delay of 0.06s can be seen clearly. The value of the delay (about 0.06s, read from the simulation) agrees with the delay calculated from TFM $F(s)$.

Figure 5.7: Disturbance $d(t)$ and its estimate $\hat{d}(t)$ ($\mu = 25$).

More calculation and simulation results are listed in Table 5.1 where the
delay is calculated and simulated with different \([\Delta A, \Delta B]\), respectively. One can see that the time delay is not affected by the value of \([\Delta A, \Delta B]\). Whatever the value of \([\Delta A, \Delta B]\), the time delay is always \(\tau_1 = 0.0605, \tau_2 = 0.0600\) second. It verifies that the variation of the plant parameter has no effects on the disturbance estimation delay \(\tau\) and the delay can be calculated by Algorithm 5.1 accurately.

<table>
<thead>
<tr>
<th>Parameter Variations</th>
<th>Delay ((\tau_i)) (second)</th>
<th>Calculated*</th>
<th>Simulated</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-0.1083, -0.0045, 0.0100, 0.0001))</td>
<td>((-0.0954, 0.1008, 0.0028, 0.000))</td>
<td>(0.060482)</td>
<td>(0.060038)</td>
</tr>
<tr>
<td>((-0.3000, 0.1000, -0.8000))</td>
<td>(0.1000, -0.0500)</td>
<td>(0.060482)</td>
<td>(0.060038)</td>
</tr>
</tbody>
</table>

*: the calculated time delay is the mean value over the frequency of interest \([0, 1]\) Hz.

**Delay compensator design**

In this subsection, Algorithm 5.2 is used to design a delay compensator \(L(s)\). The TFM \(F(s)\) has been calculated as (5.70). Since the main energy of the input signal is within the band of \([0,1]\) Hz, the pass band is set as \(\Omega_P = [0, 1]\) Hz and the stop band can be defined as \(\Omega_S = [10, \infty]\) and the transition band \(\Omega_T = [1, 10]\) Hz. Because the signal is mainly on pass band, the shapes of magnitude response over stop band and transition band are not essential. In theory, they can be of any shape.

The weighting function \(V_i(\omega), i = 1, 2, 3\) are obtained from the spectrum of the input \(u(t)\), as depicted in Figure 5.3. The weighting factors are selected as \(v_1 = v_2 = v_3 = 1\) giving the equivalent importance to magnitude approximation, phase approximation and phase delay approximation. The order of the compensator filter is chosen as 3.
The resulting optimal compensation filters are as follows:

\[
L_1(s) = \frac{1.22 \times 10^5}{s^3 + 149.4s^2 + 7436s + 1.2339 \times 10^5} \tag{5.71}
\]

\[
L_2(s) = \frac{1.248 \times 10^5}{s^3 + 149.4s^2 + 7436s + 1.234 \times 10^5} \tag{5.72}
\]

The bode plots of \(F_{ii}(s)\) and its corresponding compensation filter \(L_i(s)\), \((i = 1, 2)\) are depicted in Figure 5.8, where it can be seen that they are nearly overlapped.

![Bode plots of \(F_{ii}(s)\) and its compensation filter \(L_i(s)\)](image)

Figure 5.8: Bode plots of \(F_{ii}(s)\) (blue) and its compensation filter \(L_i(s)\) (red).

Figure 5.9 depicts the phase delay of \(F_{ii}(s)\) and its corresponding compensation filter \(L_i(s), i = 1, 2\).

**Parameter estimation with delay compensation**

By connecting the compensators \(L_i(s)\) \((5.71), (5.72)\) in a manner as shown in Figure 5.2, one can get the compensated \(u(t - \tau_i(\omega))\) and \(\hat{d}(t - \tau_i(\omega))\). The input \(u(t)\), state \(x(t)\), disturbance \((d(t)\), their estimates, and their delay compensated estimates are depicted in Figure 5.10, 5.11 respectively. It can be seen that \(\dot{d}_i(t)\) lags \(d_i(t)\) by \(\tau_i(\omega)\) second (see the top plots in Figure 5.10, 5.11 as well as Figure 5.7). By using the filter \(L_i(s)\), both the input \(u(t)\), state estimate \(\hat{x}(t)\) are lagged by \(\tau_i(\omega)\) second. Thus, all the resulting signals \([\dot{d}_i(t), \dot{x}(t - \tau_i(\omega)), u(t - \tau_i(\omega))]\) lag behind their originals simultaneously by \(\tau_i(\omega)\) second. In this way,
the delay in ARX model is compensated and the identification algorithm without delay estimation can be applied straightforward.

The results of parameter estimation are shown in Figure 5.12, where $T_I = 1$ sec, $\delta = 0.5$ sec, $T_M = 0.5$ sec, respectively. In this simulation, the parameter variation $[\Delta A, \Delta B]$ takes the value as (5.68) and takes place from 40 sec to 100 sec.

The mean values of estimated $[\hat{\Delta A}, \hat{\Delta B}]$ are

$$E[\Delta \hat{A}] = \begin{bmatrix} -0.1086 & -0.0045 \\ 0.0100 & 0.0001 \end{bmatrix}, \quad E[\Delta \hat{B}] = \begin{bmatrix} -0.0953 \\ -0.0014 \end{bmatrix}.$$  (5.73)

Compared to the actual parameter variations in equation (5.68), the proposed Algorithm 5.3 is able to give good estimates.

For comparison, Table 5.2 lists the results of Algorithm 4.1 (no delay compensation), Algorithm 5.3 (with delay compensation), with different choices of $\mu$ ($\mu = 25, 50, 100, 500, 1000$). Reading the first and second column, one can see that, the time delay varies with respect to $\mu$. As $\mu$ approaches a large value, the time delay decreases.

One can see that Algorithm 4.1 (without delay compensation) fails to give the correct estimation when $\mu$ is relatively small, (say $\mu = 25, 50, 100$). When $\mu = 1000$ ($\tau = 0.0015$sec), the results of Algorithm 4.1 is acceptable, see the last row of Table 5.2. The reason is that, when $\mu = 1000$, the time delay is so small that it can be ignored. The smaller $\mu$ is, the longer the time delay, and the worse
the performance of Algorithm 4.1. This is also the reason why we chose $\mu = 1000$ in Chapter 4.

However, by using delay compensation techniques, the proposed Algorithm 5.3 (with delay compensation) works well for all choices of $\mu$. Whatever the value of $\mu$, Algorithm 5.3 always gives the right estimation of $[\Delta A, \Delta B]$. Thus, the limitation of large $\mu$ in Algorithm 4.1 is removed giving more freedom to observer design. This is the major advantage of the proposed delay compensation approach.

**Table 5.2: Comparison of different methods**

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Delay</th>
<th>Algorithm</th>
<th>$\Delta a_{11}$</th>
<th>$\Delta a_{12}$</th>
<th>$\Delta b_1$</th>
<th>$\Delta a_{21}$</th>
<th>$\Delta a_{22}$</th>
<th>$\Delta b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>$(0.00605, 0.00000)$</td>
<td>actual</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A5.3</td>
<td>-0.1086</td>
<td>-0.0045</td>
<td>-0.0953</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A4.1</td>
<td>-0.1536</td>
<td>-0.0022</td>
<td>-0.0781</td>
<td>0.0101</td>
<td>0.0002</td>
<td>-0.0017</td>
</tr>
<tr>
<td>50</td>
<td>$(0.0301, 0.0300)$</td>
<td>actual</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A5.3</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A4.1</td>
<td>-0.1310</td>
<td>-0.0035</td>
<td>-0.0869</td>
<td>0.0101</td>
<td>0.0002</td>
<td>-0.0016</td>
</tr>
<tr>
<td>100</td>
<td>$(0.01503, 0.01500)$</td>
<td>actual</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A5.3</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A4.1</td>
<td>-0.1197</td>
<td>-0.0040</td>
<td>-0.0912</td>
<td>0.0101</td>
<td>0.0002</td>
<td>-0.0015</td>
</tr>
<tr>
<td>500</td>
<td>$(0.00300, 0.00300)$</td>
<td>actual</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A5.3</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A4.1</td>
<td>-0.1106</td>
<td>-0.0044</td>
<td>-0.0946</td>
<td>0.0101</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td>1000</td>
<td>$(0.00150, 0.00150)$</td>
<td>actual</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A5.3</td>
<td>-0.1083</td>
<td>-0.0045</td>
<td>-0.0954</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A4.1</td>
<td>-0.1094</td>
<td>-0.0045</td>
<td>-0.0950</td>
<td>0.0100</td>
<td>0.0001</td>
<td>-0.0014</td>
</tr>
</tbody>
</table>

A4.1: Algorithm 4.1 (without delay compensation)
A5.3: Algorithm 5.3 (with delay compensation)
Figure 5.10: Disturbance estimation $\hat{d}_1(t)$, and the compensated $u(t)$, $\hat{x}_1(t)$ and $\hat{x}_2(t)$ by $L_1(s)$ for identifying $\theta_1$. 
Figure 5.11: Disturbance estimation $\hat{d}_2(t)$, and the compensated $u(t)$, $\hat{x}_1(t)$ and $\hat{x}_2(t)$ by $L_2(s)$ for identifying $\theta_2$. 
Figure 5.12: The estimated parameter variations by Algorithm 5.3.
5.6 Summary

In this chapter, the disturbance estimation of high-gain observer is examined with concerns on calculating the disturbance estimation delay. For system identification, the condition of persistent excitation requires the input (exciting) signal must have enough energy over the all frequencies of interest. Due to the nonzero phase response of disturbance estimation, the time delay appears and the resulting phase-distortion challenges the parameter estimation.

The main contribution of this chapter is that the disturbance estimation delay in such a high gain observer is proved independent from the model uncertainty $[\Delta A, \Delta B]$. Hence, a novel time delay calculation algorithm is proposed and has been verified by the simulation results on a servo motor. The main benefit is the time delay calculation by this algorithm is not affected neither by the external disturbance nor by the measurement noise. This technique, on one side, improves the performance of the high-gain observer based model uncertainty estimation, and, on the other side, gives a new insight into the high-gain observer design.
Chapter 6

Robust Static Fault Detection
Observer Design

In contrast to the previous two chapters, where the observer are used to identify the parameter variations (which reflect system component faults), the next two chapters are focused on observer based actuator/sensor fault detection. The main problems to be solved are 1) how the computation burden can be reduced and 2) how the fault detection performance can be improved in a noisy environment. This chapter focuses on static observer design and its application to aero engines. By combining the frequency estimation and eigenvalue optimisation, the main contribution is the reduction of the computation complexity.

Motivated by the application of on-board fault detection to GTEs, a detailed treatment of the discrete static RFDO (Robust Fault Detection Observer) is given in this chapter. Due to the limited on-board computation resources, a fast design method is required. As most GTEs are now controlled by digital computers, a straightforward discrete observer design method is helpful to avoid discrete-continuous conversion and reduce computation costs. However, conventional $H_\infty$-norm based observers may be computation-intensive (in the context of on-board condition monitoring) and too conservative in most cases. In order to improve the fault detection performance, this chapter proposes a frequency dependent performance index with the aid of Fast Fourier Transformation (FFT)-based residual spectrum analysis. Then a left eigenvector assignment method is adopted to assign the observer poles and optimise the performance.

This chapter commences with Section 6.1 where an introduction to the on-board condition monitoring of gas turbine engine is presented, followed by a brief
review on fault detection observer design. In Section 6.2, the principles of robust static observer are presented with emphases on the eigenstructure assignment and robustness/sensitivity criteria. Section 6.3 presents a left-eigenstructure assignment based parameterisation of the static observer. In Section 6.4, a disturbance frequency estimation approach is presented and a modified robustness criterion integrating the frequency information is proposed for attenuating such a disturbance. Then, in Section 6.5, the observer is optimised by finding both optimal pole positions and values of free parameters. As illustrated in the application to the actuator fault detection of a GTE, the resulting observer shows a better performance on disturbance attenuation and fault detection.

6.1 Introduction

6.1.1 On-board condition monitoring of GTEs

In aero engines, one of the main characteristics are the dynamics between the fuel flow and shaft speeds. A general scheme, as shown in in Figure 6.1 presents the application of on-board condition monitoring to a dual-lane engine control system, where the input is the fuel flow $W_f$, the outputs are the low pressure shaft speed $N_{lp}$ and the high pressure shaft speed $N_{hp}$. The vectors $Y_s^1$, $Y_s^2$ denote the measurements of $[N_{lp}, N_{hp}]^T$ by the two set of sensors respectively. $Y_s^3$ denotes the model prediction value of $[\hat{N}_{lp}, \hat{N}_{hp}]^T$.

The control system is usually organised as a dual-lane system with two sets of parallel sensors and controllers [44]. One lane works as primary lane to issue control signals, another is waiting in "hot back-up."

An observer, a sort of mathematic redundancy, runs autonomously as the third 'virtual' lane to detect the faults in two physical lanes. The mathematical model in the third virtual lane also generates residential $\varepsilon(t)$ for detecting degradation and failures in the engine itself.

Generally, there are two categories of faults: abrupt faults and incipient faults. An abrupt fault is that a machine breaks down without any warning of impending failure (e.g., blocked filters/valves) and an incipient fault is a gradual process with a deteriorating fault condition (e.g., drift failure, deterioration of actuator, blade containments). Particularly, the earlier detection of incipient faults has a lot of benefits for reliable operation and reducing maintenance costs.
There are two classes of Luenberger observers: static observers and dynamic observers. The static observer is some proportional observer, as its feedback gain is proportional to the output estimation error via a real gain matrix. The dynamic observers are extended observers in which one or more additional terms are added in order to achieve some desired performance \[146\], \[126\]. A good introduction of robust observer based fault detection can be found in \[17\], \[147\].

In the application to on-board condition monitoring of GTEs \[44\], due to the limited computation resources, a fast RFDO (Robust Fault Detection Observer) design is required. However, the traditional \(H_2(H_{\infty})\)-norm based RFDO design is relatively high level of computation due to the fact that an integral or griding over the whole frequency range is required. Moreover, the \(H_{\infty}\) observer is designed to minimize the peaks of transfer functions at some frequency \(w_p\) for the worst-case. Note that \(w_p\) is determined by the transfer functions (system matrices), not by disturbances. Since it is more likely that the disturbance frequency \(w_d \neq w_p\), the \(H_{\infty}\) RFDO that gives the basic guarantee of the performance at the worst-case may be too conservative in some application cases.

In many industrial applications, the disturbance can be treated as a semi-stochastic process with main contents on some frequency \(w_d\). This disturbance
assumption makes sense in a lot of practical applications, such as GTEs. By optimising the performance index at \( w_d \) instead of the worst case (which requires \( H_\infty \) optimisation over the whole frequency range), the resulting observer should have a better disturbance attenuation performance.

Furthermore, in controller designs, the pole positions are determined according to the pre-specified control performance specifications \([111, 117]\). In observer designs, however, there is no an explicit way to determine the best positions of poles. Since the positions of eigenvalues affect the performance heavily, keeping eigenvalues fixed and optimising free parameters alone may not give a global optimal solution.

Keeping these two points in mind and assuming the band-limited disturbance is unknown, we proposed an approach to estimate the disturbance frequency via spectra analysis of residuals. Such frequency information is then integrated to form an improved frequency-dependent performance index for reducing the computation costs and enhancing disturbance attenuation. In the optimisation procedure, both pole positions and free parameters are optimised simultaneously. As illustrated in the simulation of a gas turbine engine fault detection, a significant improvement of disturbance attenuation is achieved compared with the existing methods. The main contribution of this chapter is to combine the frequency estimation and eigenvalue optimisation for RFDO design.

### 6.2 Problem formulation

As described above, a system like GTE corrupted by faults and disturbances can be represented in the state space form. In this chapter, a general model is considered in the discrete time domain:

\[
\begin{align*}
  x(k+1) &= Ax(k) + Bu(k) + B_f f(k) + B_d d(k) \\
  y(k) &= Cx(k) + Du(k) + D_f f(k) + D_d d(k)
\end{align*}
\]  

(6.1)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^p \), \( y \in \mathbb{R}^r \). \( f(k) \in \mathbb{R}^{n_f} \) is a general fault vector, \( B_f \) and \( D_f \) are known as fault distribution matrices. \( D_d \) are termed disturbance distribution matrices. Without loss of generality, it is assumed that the pair \( \{A, C\} \) is observable.

\( d(k) \in \mathbb{R}^{n_d} \) is a general disturbance vector due to exogenous signals, linearisation or parameter uncertainties. For instance, the disturbance caused by model
uncertainties can be presented as:

\[
d(k) = \begin{bmatrix} \Delta A x(k) + \Delta B u(k) \\ \Delta C x(k) + \Delta D u(k) \end{bmatrix}
\]  

(6.2)

In this chapter, \(d(k)\) is assumed as a quasi-stationary process with both deterministic and stochastic components:

\[
d(k) = s(k) + h(k) \ast n(k)
\]  

(6.3)

where \(s(k)\) is a band-limited deterministic disturbance vector, \(n(k)\) a white noise, \(h(k)\) the impulse response of a band-pass filter having the similar band as \(s(k)\), and \(\ast\) denotes the convolution product. Thus, \(h(k) \ast n(k)\) is a band-limited stationary stochastic signal (colored noise). It can be seen that \(d(k)\) is quasi-stationary and band-limited.

The fault distribution matrices \(B_f, D_f\) can be determined corresponding to faults to be detected. For sensor faults, they are

\[
\begin{align*}
B_f &= 0 \\
D_f &= I_r
\end{align*}
\]  

(6.4)

For actuator faults, they are

\[
\begin{align*}
B_f &= B \\
D_f &= D
\end{align*}
\]  

(6.5)

From (6.1), the TFM relating \([u(t), d(t), f(t)]\) to the output \(y(t)\) can be obtained as follows by using z-transform.

\[
y(z) = G_u(z)u(z) + G_f(z)f(z) + G_d(z)d(z)
\]  

(6.6)

where

\[
\begin{align*}
G_u(z) &= C(zI - A)^{-1}B + D \\
G_f(z) &= C(zI - A)^{-1}B_f + D_f \\
G_d(z) &= C(zI - A)^{-1}B_d + D_d
\end{align*}
\]  

(6.7)

Equation (6.1) gives the representation of a faulty system in time domain and (6.6) in frequency domain. These descriptions have been widely accepted in the fault detection and diagnosis literatures, e.g. [8], [113].
6.2.1 Structure of static observer

For system (6.1), the robust fault detection observer under consideration can be constructed by

\[
\begin{align*}
\dot{x}(k+1) &= Ax(k) + Bu(k) + Kr(k) \\
\dot{y}(k) &= C\dot{x}(k) + Du(k) \\
r(k) &= y(k) - \dot{y}(k)
\end{align*}
\]

where \( K \in \mathbb{R}^{n \times r} \) is the observer gain matrix. The observer (6.8) is the so-called static observer. Note that the gain matrix \( K \) is a constant numerical matrix and the residual \( r(k) \) is just simply amplified by \( K \) before being fed back to the observer. This indicates that the response of the observer gain on all frequencies is a constant value \( K \). Notice that the observer has \([u, y]\) as inputs and \([\hat{x}, \hat{y}]\) as outputs.

To obtain the residual equation, subtracting equation (6.8) from model (6.1) gives

\[
x(k+1) - \hat{x}(k+1) = A[x(k) - \hat{x}(k)] - K[Cx(k) + Du(k) + D_uf(k) + D_df(k) - C\dot{x}(k) - Du(k)] \\
+ B_uf(k) + B_df(k)
\]

(6.9)

Define the state estimation error \( e(k) = x(k) - \hat{x}(k) \), the estimation error and residual dynamics are governed by

\[
\begin{align*}
e(k+1) &= (A - KC)e(k) + (B_f - KD_f)f(k) + (B_d - KD_d)d(k) \\
r(k) &= Ce(k) + D_uf(k) + D_df(k)
\end{align*}
\]

(6.10)

The z-transform of (6.10) gives the TFMs relating \( r(z) \) to \( f(z), d(z) \):

\[
r(z) = \bar{G}_f(z)f(z) + \bar{G}_d(z)d(z)
\]

(6.11)

where

\[
\begin{align*}
\bar{G}_f(z) &= C(zI - A + KC)^{-1}(B_f - KD_f) + D_f \\
\bar{G}_d(z) &= C(zI - A + KC)^{-1}(B_d - KD_d) + D_d
\end{align*}
\]

(6.12)

It can be seen that, due to the existences of disturbances, the residual \( r(z) \) is not zero, even if no fault occurs. The disturbances can be a source of false and missed alarms. In order to avoid false alarms, the concept of RFDO was proposed in literatures (see [17] and its references) aiming to reduce the effects of
disturbances and enhance the effects of faults in residual $r(t)$. The RFDO problem to be solved in this chapter turns into a constrained optimisation problem:

**RFDO Design** Given a system (6.1) subject to unknown but band-limited disturbances $d(k)$, find, if possible, a real coefficient feedback gain matrix $K \in \mathbb{R}^{n \times p}$, such that the following two criteria are satisfied:

- **Stability Criterion**: The eigenvalues of $A - KC$ lie within the unit circle in $z$-plane.
- **Robustness/Sensitivity Criterion**: $\| \bar{G}_d(z) \|$ should be minimised and $\| \bar{G}_f(z) \|$ should be maximised, where $\| \cdot \|$ denotes some kind of TFMs norm.

### 6.2.2 Poles of static observer

The closed loop characteristic equation for the observer (6.8) is

$$\det(zI - A + KC) = 0$$

(6.13)

According to the definition of poles, the eigenvalue of $A - KC$ is the poles of the MIMO system (6.10), where the input is the faults $f(k)$ and disturbance $d(k)$ and the output the residual $r(k)$. The observer is stable if and only if all the eigenvalues of $A - KC$ are within the unit-circle in $z$-plane.

If the eigenvalues are chosen properly so that the $r(k)$ behaves in an asymptotically stable and adequately fast manner, then, the effects of initial condition will vanish soon and the residual $r(k)$ will tend to zero with adequate speed. In the faulty case, a non-zero residual at the steady state can be used as indication of faults.

### 6.2.3 Zeros of static observer

By observing the open loop system (6.1) in the time domain and its corresponding TFMs (6.6), (6.7) in the frequency domain, one can see that such a system consists of three subsystems:

$$
\begin{align*}
    u & \rightarrow y \\
    d & \rightarrow y \\
    f & \rightarrow y
\end{align*}
$$
Therefore, zeros of (6.1) consist of three subsets: (1) transmission zeros relating $y$ to $u$; (2) disturbance zeros relating $y$ to $d$; (3) fault zeros relating $y$ to $f$.

**Definition 6.1. Transmission Zeros of the plant** The transmission zeros of the plant (6.1) are these zeros of the TFM relating the system output $y(t)$ to the input $u(t)$. That is the set of complex numbers $z$ such that $G_u(z)$ loses rank locally

$$Z_1 = \{z| \text{rank } G_u(z) < \min(r, p)\} \quad (6.14)$$

where $r$ and $p$ are the dimensions of the output and input, respectively, and $G_u(z)$ is defined in (6.7).

Although there are several ways to define the transmission zeros of a multivariable system, the Definition 6.1 was first proposed by Rosenbrock and has been widely accepted.

Similarly, the disturbance zeros and fault zeros can be defined as follows:

**Definition 6.2. Disturbance Zeros of the plant**

$$Z_2 = \{z| \text{rank } G_d(z) < \min(r, d)\} \quad (6.15)$$

where $r$ and $d$ are the dimension of the output and the disturbance, respectively, and $G_d(z)$ is defined in (6.7).

**Definition 6.3. Fault Zeros of the plant**

$$Z_3 = \{z| \text{rank } G_f(z) < \min(r, f)\} \quad (6.16)$$

where $r$ and $f$ are the dimension of the output and the fault, respectively, and $G_f(z)$ is defined in (6.7).

**Remark 6.1.** Note that these zeros $Z_2, Z_3$ defined in (6.15), (6.16) may different from the system input transmission zeros $Z_1$ (6.14), as the TFMs $G_u(z), G_d(z)$ and $G_f(z)$ may be different from each other.

As to the zeros of the static observer (6.10), since the residual $r$ are determined by the disturbance $d$ and fault $f$, the transmission zeros of static observer (6.10) consist of two subsets: (1) disturbance zeros relating $r$ to $d$; (2) fault zeros relating $r$ to $f$. 
Definition 6.4. **Disturbance zeros of the static observer** The disturbance zeros of the static observer (6.10) are these transmission zeros of the TFM relating residual $r(t)$ to disturbance $d(t)$.

$$Z_4 = \{z|\text{rank} \tilde{G}_d(z) < \min(r, d)\} \quad (6.17)$$

where $r$ and $d$ are the dimension of the residual and the disturbance, respectively, $\tilde{G}_d(z)$ is defined in (6.12).

Definition 6.5. **Fault Zeros of the static observer** The fault zeros of the static observer (6.10) are these transmission zeros of the TFM relating residual $r(t)$ to fault $f(t)$.

$$Z_5 = \{z|\text{rank} \tilde{G}_f(z) < \min(r, f)\} \quad (6.18)$$

where $r$ and $f$ are the dimension of the residual and the fault, $\tilde{G}_f(z)$ is defined in (6.12).

Comparing the definitions between zeros of the plant and zeros of the static observer, one can see that zeros of the plant are composed of three subsets (input/disturbance/fault zeros), and zeros of the static observer consist of two subsets only (disturbance/fault zeros). This is because the dynamics of the input have been canceled in the static observer, under the condition that the model is adequately accurate.

The following theorem reveals the relationship between the observer zeros and plant zeros.

**Theorem 6.1.** In static observers, the disturbance and fault zeros are invariant. They are the same as the disturbance and fault zeros of the plant system. That is

$$Z_2 = Z_4 \quad \text{and} \quad Z_3 = Z_5 \quad (6.19)$$

**Proof.** Rewrite the plant system (6.1) as

$$\begin{align*}
x(k + 1) &= Ax(k) + [B \ B_f \ B_d] \begin{bmatrix} u(k) \\ f(k) \\ d(k) \end{bmatrix} \\
y(k) &= Cx(k) + [D \ D_f \ D_d] \begin{bmatrix} u(k) \\ f(k) \\ d(k) \end{bmatrix}
\end{align*} \quad (6.20)$$
The Rosenbrock system matrix of (6.20) is
\[
P_{\text{sys}}(z) = \begin{bmatrix} zI - A & [B \ B_f \ B_d] \\ -C & [D \ D_f \ D_d] \end{bmatrix}
\] (6.21)

It follows that the rank of system matrix (6.21) is
\[
\text{rank} P_{\text{sys}}(z) = \text{rank} \begin{bmatrix} I \ (zI - A)^{-1}[B \ B_f \ B_d] \\ -C & [D \ D_f \ D_d] \end{bmatrix} = \text{rank} \begin{bmatrix} I \ (zI - A)^{-1}[B \ B_f \ B_d] \\ 0 \ C(zI - A)^{-1}[B \ B_f \ B_d] + [D \ D_f \ D_d] \end{bmatrix} = \text{rank} \begin{bmatrix} I \ 0 \ 0 \ 0 \\ 0 \ G_u(z) \ G_f(z) \ G_d(z) \end{bmatrix}
\]

This also verifies that the transmission zeros of system (6.1) consist of three subsets: they are those values of \( z \) for which either \( G_u(z) \) or \( G_f(z) \) or \( G_d(z) \) loses rank locally. Similarly, rewriting the static observer (6.10) gives
\[
\begin{cases}
e(k + 1) = (A - KC)e(k) + [B_f - KD_f \ B_d - KD_d] \begin{bmatrix} f(k) \\ d(k) \end{bmatrix} \\
r(k) = Ce(k) + [D_f \ D_d] \begin{bmatrix} f(k) \\ d(k) \end{bmatrix}
\end{cases}
\] (6.22)

The system matrix of such a static observer and its rank turn out to be
\[
\text{rank} P_{\text{sobv}}(z) = \text{rank} \begin{bmatrix} zI - (A - KC) & [B_f - KD_f \ B_d - KD_d] \\ -C & [D_f \ D_d] \end{bmatrix} = \text{rank} \begin{bmatrix} zI - A & [B_f \ B_d] \\ -C & [D_f \ D_d] \end{bmatrix}
\]
\[ \begin{align*}
= \text{rank} \begin{bmatrix}
I & (zI - A)^{-1}[B_f B_d] \\
-C & [D_f D_d] \\
I & (zI - A)^{-1}[B_f B_d] \\
0 & C(zI - A)^{-1}[B_f B_d] + [D_f D_d] \\
I & 0 \\
0 & [C(zI - A)^{-1}B_f + D_f] [C(zI - A)^{-1}B_d + D_d] \\
I & 0 \\
0 & G_f(z) G_d(z)
\end{bmatrix}
\end{align*} \]

(6.23)

It can be seen that the transmission zeros of the static observer (6.10) consist of two subsets: \( \{ z | \text{rank} G_d(z) < \min(r, p) \} \bigcup \{ z | \text{rank} G_f(z) < \min(r, f) \} \). This proves that they are the same as the fault zeros and disturbance zeros of system (6.1).

\[ \square \]

**Remark 6.2.** Theorem 6.1 implies that a static observer does not change the disturbance zeros and fault zeros. Therefore, those zeros can not be assigned in the static observer design. It should be noted that Theorem 6.1 can be regarded as an extension of the well-known static compensator result, where zeros are invariant under proportional state/output feedback. The generalisation to fault and disturbance zeros in state observer appears to be new.

In a summary, poles of a static observer can be assigned arbitrarily by some techniques like eigenstructure assignment, but zeros are invariant. In the following sections, a RFDO design method is proposed by assigning the observer poles optimally.

### 6.3 Eigenstructure parameterisation

As discussed earlier in Section 6.2 the observer design problem in fact is an optimisation problem constrained by system stabilisation requirement. A conventional method to reach the design aims is eigenstructure assignment, which first parameterises the TFM \( \bar{G}_d(s) \) and \( \bar{G}_f(s) \) and the corresponding gain matrix \( K \) into two sets of parameters:

- a set of closed-loop eigenvalues \( \{ \lambda_i \} \);
- a set of free parameter vectors \( \{ q_i \} \)
In this section, the aim is to parameterise the TFMs of the static observer in terms of eigenstructure, rather than assign poles to desired places directly. Indeed, the poles are determined by optimisation programme. Before presenting the approach, the following assumption should be introduced which is used throughout:

**Assumption A1.** The poles \( \{\lambda_i\} \) of the closed loop observer (6.8) are distinct from those of the open loop plant system (6.1), that is,

\[
|\lambda_i I - A| \neq 0, \quad i = 1, 2, \ldots, n
\]

This assumption makes \((A - \lambda_i I)^{-1}\) exist. It should be pointed out that this assumption is used just for simplifying the presentation of the eigenstructure assignment and avoiding the inverse of a singular matrix. In the case where an open-loop eigenvalue appears in the closed-loop eigenvalue set, a slight modification is needed, e.g., a numerical algorithm of singular value decomposition (SVD). For more details, please refer to [22], [117].

If assumption A1 holds, derived from [17], [22], [117], [118], the parametric expression of the gain matrix \( K \) can be presented as:

**Lemma 6.1.** Let \( \{A, C\} \) be observable, then, for any group of scalars \( \lambda_i, i = 1, 2, \ldots, n \) under assumption A1, the gain matrix \( K \) can always be parameterised as:

\[
K = L^{-1}Q
\]

(6.24)

where \( L \in \mathbb{R}^{n \times n} \) is composed of the left eigenrows \( l_i \) of \( A - KC \)

\[
L = \begin{bmatrix}
l_1^T \\
\vdots \\
l_n^T
\end{bmatrix} = \begin{bmatrix}
q_1^T C (A - \lambda_1 I)^{-1} \\
\vdots \\
q_n^T C (A - \lambda_n I)^{-1}
\end{bmatrix}
\]

(6.25)

and \( Q \in \mathbb{R}^{n \times r} \) consists of a set of free parameter vectors \( q_i \)

\[
Q = \begin{bmatrix}
q_1^T \\
q_2^T \\
\vdots \\
q_n^T
\end{bmatrix}
\]

(6.26)
Lemma 6.1 gives an explicit, parametric expression of $K$, with the eigenvalues $\lambda_i$ and the vectors $q_i$ as the free parameters. Except for the assumption $A1$ and the following two constraints, the eigenvalues $\lambda_i$ can arbitrarily be chosen from the fields $\mathbb{C}$ of real (or complex) numbers, and free parameters $q_i$ from the real vector spaces $\mathbb{R}^p$, respectively. In RFDO, the freedom allows the performance criterion satisfied by selecting appropriate values of $\lambda_i$, $q_i$. These two constraints are:

**Constraint C1.** The set $\{\lambda_i\}$ and the set $\{q_i\}$ of free parameter vectors must be self-conjugated such that the resulting feedback matrix $K$ is a real coefficient matrix. That is

\[
\begin{align*}
(1) \quad & \forall \lambda_i \in \Lambda, \exists \lambda_i^* \in \Lambda \\
(2) \quad & \forall q_i \in Q, \exists q_i^* \in Q \\
(3) \quad & \forall \lambda_i \text{ and its corresponding vector } q_i \exists q_i^* \text{ which is the corresponding vector of } \lambda_i^*
\end{align*}
\]

where $\Lambda$ presents the set of closed-loop eigenvalues $\Lambda := \{\lambda_i\}_i = 1, 2, \ldots, n$.

**Constraint C2.** $\lambda_i$ and $q_i$ must be chosen in such a way that $l_i$ yields linearly independent vectors. Then left eigenvector matrix $K$ is non-singular and the inverse $K^{-1}$ exists.

As a result, $\bar{G}_f(z)$, $\bar{G}_d(z)$ can be expressed in a parametric form.

**Lemma 6.2.** Under the assumption $A1$ and constraints $C1$, $C2$, the discrete TFMs $\bar{G}_d(z)$, $\bar{G}_f(z)$ (6.12) can be expanded in terms of the eigenstructure as

\[
\begin{align*}
\bar{G}_d(z) &= D_d + C \left[ R \Psi(z) L \right] (B_d - L^{-1} Q D_d) \\
\bar{G}_f(z) &= D_f + C \left[ R \Psi(z) L \right] (B_f - L^{-1} Q D_f)
\end{align*}
\]

(6.27)

respectively, where

\[
\Psi(z) = \begin{bmatrix} \frac{1}{z-\lambda_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{z-\lambda_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{z-\lambda_n} \end{bmatrix}
\]

(6.28)

and

\[
R = L^{-1} = (r_1, r_2, \ldots, r_n)
\]

(6.29)
Proof. Similar to the Lemma 10.2 in [22], by replacing $s$-transform by $z$-transform, the inverse of any discrete TFM $(zI - A + KC)^{-1}$ can be expanded as

$$(zI - A + KC)^{-1} = \frac{r_1 l_1^T}{z - \lambda_1} + \frac{r_2 l_2^T}{z - \lambda_2} + \ldots + \frac{r_n l_n^T}{z - \lambda_n} \quad (6.30)$$

Substituting (6.30) into (6.12) gives the parametric expressions of $G_d(z)$ and $G_f(z)$. Rewriting it into matrix form, (6.27) is thus given. \hfill \square

It is worth to note that all the vectors $l_i$, $r_i$ (matrices $L$, $R$) depend on the choice of $\lambda_i$ and $q_i$ ($i = 1, 2, \ldots, n$). The parametric expressions of $\bar{G}_d(z)$ and $\bar{G}_f(z)$ allows one to use a lot of existing optimisation algorithms to find the optimal gain matrix $K$. Before optimisation, an issue needed to be addressed is the definition and evaluation of the robustness/sensitivity criteria and related TFM-norms

### 6.4 Frequency dependent performance index

One of the widely accepted robustness/sensitivity criteria is the $H_{\infty}$-norm based index: $J_{\infty/\infty} = \frac{\|G_d(z)\|_{\infty}}{\|G_f(z)\|_{\infty}}$, where $H_{\infty}$-norm is used to measure the largest singular value of a TFM over the whole frequency range. Minimising $J_{\infty/\infty}$ is to find a matrix $K$ so that the fault detection is optimal at the worst case. However, the $H_{\infty}$ optimal observer may be too conservative, because it only minimises the peak value and gives the basic guarantee. A further drawback is the computation complexity. $H_{\infty}$-norms are calculated by gridding and computing over the whole frequency range $0 \leq |\omega| \leq \pi$. The computation of $H_{\infty}$ calculation is too intensive to be afforded by the embedded computer of GTEs. In order to avoid these drawbacks, in this study, a stable observer is first set up to generate a sequence of residuals and the Fast Fourier Transform (FFT) technique is used to estimate the disturbance frequency. Such frequency information is then incorporated into the robustness/sensitivity performance indices to form a modified performance criterion.

#### 6.4.1 Disturbance frequency estimation

As stated in the problem formulation, in our study, the disturbance is unknown except it is assumed band-limited. In order to improved the disturbance attenuation performance, it is essential to estimate the unknown disturbance frequency
The following theorem is used to solve the problem of disturbance frequency estimation.

**Theorem 6.2.** For a multi-variable discrete system (6.1), under the assumption that the disturbances is frequency band-limited, if the observer (6.10) is stable, at steady state, the main spectrum set of residual, $\Omega_r$, is limited to a portion of $\Omega_d$ (which is the spectrum set of disturbance). That is

$$\Omega_r \subseteq \Omega_d$$

**Proof.** Without loss of generality, the disturbance is supposed to be with the complex waveform vector $d_0(k) \in \mathbb{C}^{n_d}$ at some frequency $w_0$

$$d_0(k) = A_0 e^{jw_0 k} \quad (6.31)$$

where $A_0 = [a_{0,1} \ a_{0,2} \ldots a_{0,n_d}]^T \in \mathbb{R}^{n_d}$ is the magnitude vector and $n_d$ the dimension of the disturbance vector. Assume that the sequence length is $N$, when $N$ is large enough, a rational frequency $w_0$ can be expressed as

$$w_0 = 2\pi r/N, \quad (r \leq N), \quad (6.32)$$

then the N-point FFT of $d_0(k)$ is

$$D[n] = \frac{1}{N} \sum_{k=0}^{N-1} A_0 e^{jw_0 k} \cdot e^{-j2\pi nk/N}$$

$$= A_0 \delta[n - r] \quad (0 \leq n \leq N)$$

where $D[n] \in \mathbb{C}^{n_d}$ and $\delta[n - r]$ is the Dirac delta function. Defining $0_{n_d \times 1} = [0 0 \ldots 0]^T \in \mathbb{R}^{n_d}$, $D[n]$ can be written as

$$D[n] = \begin{cases} A_0 & \text{when } n = r; \\ 0_{n_d \times 1} & \text{otherwise;} \end{cases}$$

It can be seen that the frequency contents of $d_0(k) = e^{jw_0 k}$ is mainly at $n = r$ (corresponding to the digital frequency $w_0 = 2\pi r/N$) with magnitude $A_0$.

Let $\tilde{G}_d(z) \in \mathbb{C}^{n_r \times n_d}$ denote the TFM relating residual $r(k) \in \mathbb{C}^{n_r}$ to disturbance $d_0(k) \in \mathbb{C}^{n_d}$, where $n_r$ is the dimension of the residual, it follows that the
N-point FFT of $r(k)$ is

\[
\mathbf{R}[n] = \mathbf{G}_d(e^{j2\pi n/N}) \cdot \mathbf{D}[n] = \begin{cases} 
\mathbf{G}_d(e^{j2\pi r/N}) \mathbf{A}_0 & \text{when } n = r \\
0_{n_r \times 1} & \text{otherwise}
\end{cases}
\]

The (power) spectrum of $r(k)$ is defined as the sum of $|\mathbf{R}_r[n]|$ squared. That is

\[
\|\mathbf{R}[n]\|^2 = \sum_{i=1}^{i=n_r} |R_i[n]|^2 = \sum_{i=1}^{i=n_r} R_i^*[n]R_i[n] = (\mathbf{R}^*[n])^T \cdot \mathbf{R}[n] = \begin{cases} 
\mathbf{A}_0^T \mathbf{G}_d^*(e^{j2\pi r/N})^T \cdot \mathbf{G}_d(e^{j2\pi r/N}) \mathbf{A}_0 & \text{when } n = r \\
0 & \text{otherwise}
\end{cases}
\]

Hence, if $\|\mathbf{G}_d(e^{j2\pi r/N})\| \neq 0$, the spectrum of $r(k)$ achieves nonzero value at disturbance frequency $w_0 = 2\pi r/N$ and null at all other frequency points. That is

\[
\Omega_r = \{\omega_0\}
\]

According to the theory of Fourier series, for any sort of disturbance $\mathbf{d}(k)$, $\mathbf{d}(k)$ can be treated as a linear combination of a set of vectors

\[
\sum_i \mathbf{d}_i(k) = \sum_i \mathbf{A}_i e^{jw_i k}.
\]

According to the linearity of LTI system, the residual corresponding to $\sum \mathbf{d}_i(k)$ is the sum of the residuals corresponding to individual disturbance $\mathbf{d}_i(k)$. Therefore, the spectrum of residuals corresponding to $\mathbf{d}(k)$ is nonzero at disturbance frequency $w_i$ and null at other frequency points.

Hence, if a disturbance $\mathbf{d}(k)$ contains component at frequency $w_i$ and $\|\mathbf{G}_d(jw_i)\| \neq 0$, then the corresponding residual also have contents at $w_i$. It follows that

\[
\Omega_r \subseteq \Omega_d
\]
where
\[ \Omega_r = \{ \omega_n | \omega_n = 2\pi n / N, ||R[n]||^2 \neq 0 \} \quad (6.36) \]
and
\[ \Omega_d = \{ \omega_n | \omega_n = 2\pi n / N, ||D[n]||^2 \neq 0 \} \quad (6.37) \]

Theorem 6.2 is an extension of the SISO result to univariate system. The extension to multivariable system and quasi-stationary signal appears to be new. It can be simply interpreted as: for a discrete-time observer, under the condition of stabilization, the disturbance frequency \( w_d \) of \( d(k) \) has not changes and then can be identified from residual \( r(k) \).

**Remark 6.3.** It is worth to note that the spectrum set of residuals is just a part of the spectrum set of disturbances. For some disturbance \( e^{j\omega_0 k} \), if the magnitude \( ||G_d(z)||_{z = e^{j\omega_0}} = 0 \), then \( \omega_0 \) does not appear in \( \Omega_r \). It means the residual \( r(k) \) is not affected by the disturbance \( d(k) \). Hence, it is not necessary to attenuate such a disturbance.

### 6.4.2 Robustness Index

The basic idea behind RFDO is to minimise the effects on residual \( r(k) \) from the disturbance \( d(k) \) and maximise the effects from the fault \( f(k) \). From the viewpoint of optimal (disturbance) decoupling, the *robustness index* measures of the effects from \( d(k) \) to \( r(k) \) and *sensitivity index* measures the effects from \( f(k) \) to \( r(k) \).

As discussed earlier, the \( H_\infty \)-norm based index has two drawbacks: (a) the computation cost is too high to be applicable for the on-board fault detection; (b) \( H_\infty \) may be too conservative.

Due to the fact that the unknown disturbance frequency can be estimated from the residual spectrum (see Theorem 6.2), a modified robustness index is proposed to relieve the two limitation of \( H_\infty \)-norm based observer. The concepts behind the proposed index are to integrate the disturbance frequency information and make the resulting observer optimal for such a certain disturbance.

The proposed robustness index is to evaluate the TFM-norm of \( \bar{G}_d(z) \) at the
disturbance frequency point \( z = e^{jw_d}, \omega_d \in \Omega_d \),

\[
\min_{Q, \Lambda} \| \tilde{G}_d(z) \|_{z = e^{jw_d}} \tag{6.38}
\]

rather than the whole frequency range. Note that, in the discrete frequency domain, \( 0 \leq |w_d| \leq \pi \).

With the aid of Theorem 6.2, the unknown disturbance frequency \( \Omega_d \) can be estimated as \( \Omega_r \). It follows that, if the spectrum of the residual mainly lies at frequency range \( \Omega_r \), then (6.38) can be computed equivalently by

\[
\min_{Q, \Lambda} \{ J_1 = \| \tilde{G}_d(z) \|_{z = e^{jw_r}} \} \tag{6.39}
\]

where \( w_r \in \Omega_r \).

By minimising the robustness index (6.39) at the frequency band \( \Omega_r \), the residual is designed maximally robust to the disturbances.

### 6.4.3 Sensitivity Index

Not like a random noise, a fault signal is associated with some pattern, and, in the frequency domain, its distribution is not uniform over the whole frequency range. For instance, an incipient fault comprises mainly low frequency components. For abrupt faults, high frequency contents only exist at the time instant when faults start, and it is almost constant (zero frequency) content thereafter. For detecting these common faults with main contents on low frequency, the steady state gain is the most important factor. Hence, [17] proposed strong fault detectability condition: \( \| G_f(s) \|_{s=0} \neq 0 \) in continuous time domain. In our study, it is proposed that \( \| G_f(z) \|_{z=1} \) index should be maximised for magnifying the fault signal to the greatest extent, which gives

\[
\max_{Q, \Lambda} \{ J_2 = \| \tilde{G}_f(z) \|_{z=1} \} \tag{6.40}
\]

Generally, for a fault with main frequency components at discrete frequency \( w_f \) (which is known a priori), the sensitivity index can be defined as

\[
\max_{Q, \Lambda} \{ J_2 = \| \tilde{G}_f(z) \|_{z = e^{jw_f}} \} \tag{6.41}
\]

Combining the robustness index (6.39) and sensitivity index (6.41) leads to
the performance index as:

\[
\min_{Q,A} \{ J = \frac{J_1}{J_2} = \frac{\| \bar{G}_d(z) \|_{z=e^{w_r}}} {\rho + \| G_f(z) \|_{z=e^{w_f}}} \} \tag{6.42}
\]

where \( \rho \) is a small positive real number. The aim is to avoid division by zero in case of \( \| G_f(e^{w_f}) \| \) is zero.

**Remark 6.4.** Since the variables \( z \) in (6.42) are given specific values \( w_r \) or \( w_f \), respectively, the computation of the TMF-norm is converted into a numerical matrix norm calculation. Compared to the \( H_\infty \) TMF-norm, which requires gridding, computing and finding the largest singular value over the whole frequency \([0, \pi]\), performance index (6.42) only involves the computation of two real numerical matrices. The associated computation is very low.

### 6.5 Optimisation of free parameters and eigenvalues

It can be seen from (6.27), the performance function \( J \) (6.42) is not only a function of the free parameters \( \{q_i\} \), but also a function of the eigenvalues \( \{\lambda_i\} \). The eigenvalues determine the stability, and affect the performance index to a great extent. In most papers, however, only \( \{q_i\} \) are optimised, the eigenvalues are given a prior. In practice, the specification of desired place for pole assignment is done in a trial-and-error manner. Thus, pole position may not be optimal and the resulting solution is likely local optimal [111].

One way to improve the disturbance rejection performance is to optimise both \( A \) and free parameters \( Q \) simultaneously. In the proposed optimisation approach, no exact positions of poles are pre-specified, whereas the regions at which poles may locate are specified. These regions are determined roughly according to the stability and response speed requirement, but the final pole positions are determined by the optimisation.

In order to ensure both close-loop stability and dynamical performance are met, the poles \( \{\lambda_i\} \) are required to meet the following specification:

\[
\lambda_i \in \chi_i, i = 1, 2, \ldots, n \tag{6.43}
\]

where \( \chi_i, i = 1, 2, \ldots, n \) are a group of simply connected regions all within the
unit circle on the $z$-plane. Generally, for a real eigenvalue, $z_i$, choose

$$
\chi_i = [\alpha_i^l \alpha_i^u],
$$

(6.44)

where the superscript $l$ denote the lower boundary and $u$ the upper boundary.

For a pair of complex eigenvalues, $z_j$ and $z_k (= z_j^*)$, choose

$$
\chi_j = \{s = \alpha_j + j\beta_k | \alpha_j \in [\alpha_j^l \alpha_j^u], \beta_k \in [\beta_k^l \beta_k^u]\}
$$

(6.45)

and

$$
\chi_k = \{s = \alpha_j - j\beta_k | \alpha_j \in [\alpha_j^l \alpha_j^u], \beta_k \in [\beta_k^l \beta_k^u]\}
$$

(6.46)

where $\alpha_j^l, \alpha_j^u, \beta_k^l$ and $\beta_k^u$ are real number specifying the boundary of the regions. It is worth to note that the conjugated poles $z_j$ and $z_k$ leads to self-conjugated $\chi_j$ and $\chi_k$. $\chi_j$ and $\chi_k$ shares same parameters $\alpha_j^l, \alpha_j^u, \beta_k^l, \beta_k^u$.

As to the selection of the pole regions, an intuitive rule is that they should be far from the disturbance frequency and close to the fault frequency. The reason is that, if $z_i$ is a pole of a system, it makes the magnitude response in the neighborhood of $z_i$ very large. In order to reduce the magnitude response of the disturbance, it is useful to have the pole being far from the disturbance frequency. On the other hand, placing these pole regions close to the fault frequency would help to amplify the effects of the fault.

Based on the discussion above, the solution to the RFDO can now be stated as follows:

*If assumption A1 and constraints C1, C2 are satisfied, and the main frequency contents of residuals $r(k)$ can be estimated at $w_r$ given by Theorem 6.2, then minimising the following performance index

$$
J(Q, \Lambda) = \frac{\sum_{\omega_r \in \Omega_r} \left| D_d + C R \Psi(z) L (B_d - L^{-1} Q D_d) \right|_{z = e^{j\omega_r}}}{\rho + \sum_{\omega_f \in \Omega_f} \left| D_f + C R \Psi(z) L (B_f - L^{-1} Q D_f) \right|_{z = e^{j\omega_f}}} \quad (6.47)
$$

* gives the optimal gain matrix $K = L^{-1} Q$ such that the disturbance $d(k)$ is attenuated and the sensitiveness to faults is enhanced to the greatest extent.*
6.6 Application and results

To illustrate the proposed RFDO design approach, this section presents the results of an application to the detection of actuator faults of a gas turbine engine. Real engine fuel flow data collected from normal engine closed-loop operation at the engine test-bed are used [44].

In this application, the main concern is the detection of actuator faults, and both sensors are assumed fault-free. Abrupt faults and incipient faults are considered here. The GTE model is expressed in the state space form

\[
\begin{align*}
\begin{bmatrix}
x_1(k+1) \\
x_2(k+1)
\end{bmatrix} &= \begin{bmatrix}
0.9769 & 0.0038 \\
0.0936 & 0.9225
\end{bmatrix}
\begin{bmatrix}
x_1(k) \\
x_2(k)
\end{bmatrix} + \begin{bmatrix}
2.1521 \\
8.8186
\end{bmatrix} W_f(k)
\end{align*}
\]

with the sampling interval \( T_s = 0.025 \) second. It is easy to verify that the system (6.48) is observable and the open-loop poles are \([0.9828, 0.9166]\). The disturbance model is assumed as

\[
\begin{align*}
B_d &= \begin{bmatrix}
0.1510 & 0.0406 \\
0.0500 & 0.0528
\end{bmatrix}, \\
D_d &= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \\
B_f &= B = \begin{bmatrix}
2.1521 \\
8.8186
\end{bmatrix}, \\
D_f &= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\end{align*}
\]

(6.49)

where the fault matrix \( B_f = B \) simulating actuator faults. The disturbance injected into the system is simulated by

\[
\begin{align*}
\begin{bmatrix}
d_1(k) \\
d_2(k)
\end{bmatrix} &= \begin{bmatrix}
s_1(k) + h(k) * n_1(k) \\
s_2(k) + h(k) * n_2(k)
\end{bmatrix}
\end{align*}
\]

(6.50)

where \([n_1(k), n_2(k)]\) are white noises with covariance matrix \( \begin{bmatrix}
0.4 & 0 \\
0 & 0.04
\end{bmatrix} \) and zero mean values, \( h(k) \) is a filter with pass band of \([1.5, \pi]\). The deterministic
signals are

\[
\begin{align*}
    s_1(k) &= \sin(2k) + \cos(2.1k + \frac{\pi}{4}) + 0.5\cos(2.3k) + 0.5\sin(2.2k - \frac{\pi}{4}) \\
    s_2(k) &= 0.5\sin(2.1k) + 0.25\sin(1.5k + \pi/4)
\end{align*}
\] (6.51)

Figure 6.2: Disturbances injected into the system

Figure 6.2 depicts the disturbance in the time domain, and its 128-point FFT based spectrum is shown in Figure 6.4(a).

In order to estimate the disturbance frequency, a stable observer is first constructed and its gain matrix $K_0$ is computed via Matlab command \textit{place}(A',C',[0.5 0.5]')'. The input is depicted in Figure 6.3 (a), and the two outputs (high/low pressure shaft speed [$N_{lp}, N_{hp}$]) and their estimates given by $K_0$ are depicted in Figure 6.3 (b) and (c), respectively. It can be seen that, the estimation error is very small, because of the high model quality. A further reason is that the output values of [$N_{lp}, N_{hp}$] is relatively large, and the measurement sensor noise is not obvious.

An 128-points FFT is employed to calculate the spectrum of $r(k)$, as shown in Figure 6.4 (b). Compared to the spectrum of disturbances (Figure 6.4 (a)), it can be seen that $r(k)$ is a band-limited quasi-stationary signal and has the main frequency components around $w_d = 2.1$ \textit{rad/sample}.

By considering the effects of poles to the magnitude response, the desired poles should keep distance from the disturbance frequency points ($z = e^{\pm 2.1j}$). And for the sake of simplicity, the desired poles are confined at real axis. Thus,
the desired pole regions are set as follows: $[\alpha^l_i, \alpha^u_i] = [-0.75, 0.75]$, $i = 1, 2$, and $[-0.5, 0.5]$ are set as the initial values of $\lambda_i$. The initial value of $Q$ are

$$Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.1 \\ 1.0 & 0.0 \end{pmatrix}$$

(6.52)

It is easy to verify that the assumption $A1$ and constraints $C1$, $C2$ are satisfied. The value of $\rho$ in (6.42) is set as 0.1.

Using $fmincon$ function provided by MATLAB Optimisation Toolbox gives the optimal gain matrix $K$

$$K_{opt} = \begin{pmatrix} 3.3769 & -11.0584 \\ 0.9516 & -2.8394 \end{pmatrix}$$

(6.53)

with optimal poles $[0.6519, 0.7100]$ and $J = 0.0005583$. For comparison, $K_{place}$ (by $place$ command provided by Matlab) and $K_{inf}$ (by $H_\infty$ method) are designed,

$$K_{place} = \begin{pmatrix} 0.3250 & 0.0038 \\ 0.0936 & 0.2125 \end{pmatrix}, \quad K_{inf} = \begin{pmatrix} 0.2402 & 0.0549 \\ 0.0493 & 0.2973 \end{pmatrix}$$

(6.54)
where $K_{\text{place}}$, $K_{\text{inf}}$ have identical eigenvalues as $K_{\text{opt}}$.

### 6.6.1 Residuals without faults

Figure 6.5 shows the residuals $\|r(k)\|_\text{opt}$, $\|r(k)\|_\text{place}$, $\|r(k)\|_\text{inf}$ and their spectra, respectively. The disturbance attenuation of $K_{\text{opt}}$ is more apparent compared to that of $K_{\text{place}}$, $K_{\text{inf}}$. In the time domain, it can be seen that the maximum magnitude of $\|r(k)\|_\text{opt}$ is below 3 in the steady state, however that of $\|r(k)\|_\text{place}$ and $\|r(k)\|_\text{inf}$ are nearly 4.

In the frequency domain, the disturbance attenuation of $K_{\text{opt}}$ is obviously better. Particularly, the residual spectrum magnitude of observer $K_{\text{opt}}$ at $\omega_d = 2.1$ rad is attenuated to 20. The performance of disturbance attenuation is expressed by the ratio of the power of $d(k)$ and $r(k)$ in decibel (dB), as shown

$$\text{dB} = 10\log_{10} \frac{\|R(j\omega)\|^2}{\|D(j\omega)\|^2}$$

(6.55)

At frequency 2.1 rad, the disturbance is attenuated -8 dB by $K_{\text{opt}}$. Whereas it is 0 dB in $K_{\text{place}}$ and $K_{\text{inf}}$. The benefit of the smaller residual amplitude of $\|r(k)\|_\text{opt}$ is that $K_{\text{opt}}$ is able to detect a smaller fault and avoid many false alarms.

### 6.6.2 Detection of actuator faults

Although many actuator faults lead to an abrupt changes, in practice, actuator faults can also be caused by the components degradation and behave as slow
changes. Such faults are extremely difficult to be detected immediately from a simple visual inspection of the output signals. To simulate the incipient fault of the fuel pump gain drift of 0.002 unit per second, the fault function $f(t)$ is represented as

$$f(t) = \begin{cases} 
0 & (t \leq 10.05) \\
0.002(t - 10.05) & (10.05 < t < 20.05) \\
0.008 & (t \geq 20.05) 
\end{cases}$$  \hspace{1cm} (6.56)

This is a typical saturated actuator fault caused by component degradation. Figure 6.5 shows the norms of the residual vectors. The observers $K_{\text{place}}$, $K_{\text{inf}}$ fail to detect such a fault, as there is no obvious changes in their residuals. However, $\|r(k)\|_{\text{opt}}$ shows an increase soon after the fault happening, and then follows the fault. From the view point of fault detection delay, it is less than 5 second after the fault happening when $K_{\text{opt}}$ gives fault indication and no false alarm thereafter. However, even 20 seconds later, both $K_{\text{place}}$, $K_{\text{inf}}$ fail to detect the fault. This verifies that $K_{\text{opt}}$ is able to detect an incipient fault earlier and more distinctively.
Figure 6.6: Residuals of $K_{opt}$, $K_{place}$ and $K_{inf}$ in the case of a saturated incipient actuator fault.

### 6.7 Summary

In this chapter, a robust fault detection observer has been designed through FFT-based disturbance frequency estimation and left eigenvector-based eigenstructure assignment. The disturbance frequency is estimated by using the spectrum analysis of residuals generated from any stable observer. Furthermore, both the free parameters $\{q_i\}$ and the eigenvalues $\{\lambda_i\}$ are optimised simultaneously.

The advantage is the reduction of the computation complexity. By integrating disturbance frequency information into the proposed performance index, the index evaluation involves the calculation of numerical matrix norm rather than TFM norm. With the aid of FFT, the evaluation of the new performance index demands much less computation than $H_\infty$-norm based indices. The modified performance index can be seen as some kind of constrained $H_\infty/H_\infty$ performance index, as a constraint of band-limitation is added to the disturbances.
In Chapter 6 the observer is confined to the classic (static) Luenberger structure, where the gain is a constant numerical matrix without the ability of frequency shaping \[108, 149, 150, 122\] and the zeros are invariant. In order to add more design freedom in the robust fault detection observer design, with the inspiration from the dynamic output feedback control design, an extended observer (termed as dynamic observer) is considered. The proposed dynamic observer has some desirable frequency shaping characteristics due to its dynamic feedback gain. The new structure retains all of the characteristics of the static observers and, in addition, provides additional degrees of freedom in the design which can be used to improve the disturbance attenuation performance.

Different from all the reported results on static/dynamic observer design, our study aims to establish a zero assignment approach in dynamic observer design and provide systematic study of its properties. It will be shown that the dynamic feedback gain of a dynamic observer introduces additional zeros, and both the observer poles and the additional zeros can be assigned arbitrarily. Hence, the zero assignment is proposed to locate the zeros at disturbance frequencies, and the poles can be selected by optimisation procedure. Note that the zero assignment is possible only in dynamic observers. The zero assignment in multivariable dynamic observer design would be the main contribution.

This chapter is mainly based on the use of continuous-time system models, although the techniques developed can be directly applied to discrete-time models. This chapter is organised as follows: The properties of observer zeros, the
possibility of zeros assignment are studied in Section 7.2 and 7.3. In Section 7.4, a detailed design procedure is given. An application to fault detection of a multi-variable system and its results are illustrated in Section 7.5 where the faults have the same distribution matrix as the disturbance. In this context, the $H_\infty/H_\infty$ static observer design (without weighting functions) may fail, as the TFMs relating the residual to the fault and the disturbance are identical and the $H_\infty/H_\infty$ performance index is always equal to 1. As shown in the simulation, by using dynamic observer, a better noise attenuation and fault detection performance have been obtained.

7.1 Introduction

The fault detection observer problem can be formulated as an estimation problem where system outputs are estimated and certain robustness and sensitivity performance must be satisfied. As depicted in Figure 7.1, the fault detection observer composes of two parts: (1) forward path (in fact, it is the model of dynamic system); (2) feedback path (the so-called observer gain or feedback gain). From the viewpoint of Luenberger observer design, the forward path (system model) is given from the plant model, and the observer design is to select a good feedback gain.

![Figure 7.1: Structure of the observer-based fault detection](image)

Most of existing fault detection observers have been simply confined in traditional static observers \[15, 17, 21, 14, 22, 122\]. Here, the term static observer is used to denote the classic Luenberger observer, in which a constant numerical gain is used in the feedback path to process the residual signal. It is well known that the non-unique solution to the gain matrix brings the freedom to design an optimal observer. Although the static observer is able to shift the
poles only, its zeros are invariant. As the performance of an observer depends not only on poles, but also on positions of zeros, the zero invariance imposes a limitation on the static observer design.

Therefore, it is a natural idea to introduce additional dynamics into observers for a better performance [151], [122]. In order to distinguish from classic observers, the term dynamic observer is used, where a dynamic system is employed to replace the constant gain matrix in the feedback path. Comparing to the static observers with only one gain matrix, dynamic observers provide more design freedom, and presents both advantages and challenges.

Some preliminary works have been done on dynamic observers paying attention mainly to the poles assignment. PIO (Proportional Integral Observer) and PMIO (Proportional Multiple Integral Observer) are discussed in [123], [125]. Input-output observer with dynamic feedback is proposed in [150] and [122]. In [128], a dynamic observer design method is proposed as a dual of control design for the state estimation. A similar work is the Lipschitz UIO (Unknown Input Observer) [108], where two dynamic compensators are introduced to tackle Lipschitz nonlinearities. It is worth noting that, all the reports on dynamic observer design concern the poles and ignore the additional zeros introduced by the dynamic feedback. The extra free parameters provided by dynamic observers are determined roughly by optimisation algorithms.

From the viewpoint of system performance, the poles determine the system stability, but they are insufficient for achieving an optimal performance. It is felt that the advantages of taking zeros into account would be twofold: 1) it is more possible to attenuate the disturbance further if the observer zeros are close to the disturbance frequency; 2) the specification of zeros diminishes the search space making the computation burden reduced.

7.2 Problem formulation

This chapter is mainly based on the use of continuous-time system models, although the techniques developed can be directly applied to discrete-time models.
Consider a completely controllable and observable continuous LTI system corrupted by faults and disturbances [17]:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + B_ff(t) + B_dd(t) \\
y(t) &= Cx(t) + Du(t) + D_ff(t) + D_dd(t)
\end{align*}
\] (7.1)

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^p \) the input, \( y \in \mathbb{R}^{nr} \) the output (\( nr \geq p \)) and \( A, B, C, D, B_d, B_f, D_d \) and \( D_f \) are known with corresponding dimensions. \( d(t) \in \mathbb{R}^{nd} \) is a general disturbance vector due to exogenous signals, linearisation or parameter uncertainties. As discussed in section 6.2, without loss of generality, \( d(t) \) is assumed as a quasi-stationary process with both deterministic and stochastic components:

\[
d(t) = s(t) + h(t) * n(t)
\] (7.2)

It has been shown that \( d(t) \) is quasi-stationary and band-limited. For more details, please see section 6.2.

As discussed in Chapter 6, the most common observer technique used for fault detection is the (static) observer, see (6.8). From the viewpoint of the observer structure, as shown in Figure 7.1, the feedback path of the static observer (6.8) consists of a real matrix \( K \). In terms of frequency response, the TFM of such a feedback path is a constant value (equal to \( K \)) over the whole frequency range. Thus the observer gain only adjusts the magnitude of the residual to correct the observer behaviour, and the frequency characteristics of the residual do not change when it is fed into the forward part.

Now we extend the static structure (6.8) to a more general dynamic framework. A dynamic system is introduced to replace the constant numerical gain matrix, so that the feedback path has the ability to shape the frequency characteristics of the residual.

For system (7.1), a \( m \)-th-order dynamic observer will be used throughout this chapter with

\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + v(t) \\
\hat{y}(t) &= C\hat{x}(t) + Du(t)
\end{align*}
\] (7.3)

where \( v(t) \) is obtained from the feedback path by applying a dynamical compensator

\[
\begin{align*}
\dot{\xi}(t) &= K_1\xi(t) + K_2r(t) \\
v(t) &= K_3\xi(t) + K_4r(t)
\end{align*}
\] (7.4)
Here, $\xi \in \mathbb{R}^m$ is the \textit{dynamic feedback state vector}, $v \in \mathbb{R}^n$ the output of dynamic feedback, and

$$r(t) = y(t) - \hat{y}(t)$$  \hspace{1cm} (7.5)

the residual signal. For the sake of notation, the dynamic feedback system (7.4) may be represent in a matrix form:

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$  \hspace{1cm} (7.6)

One can see that the dynamic observer has the similar forward model (7.3) as the classic static observer. The obvious difference between the static observer and the dynamic observer is the feedback path: the real coefficient constant gain matrix $K$ in static observers is replaced with a dynamic system (7.4). This new dynamic system (7.4) offers more freedom that will be used to assign zeros. The TFM of the dynamic system (7.4) relating $r$ to $v$ is given by

$$H(s) = K_3(sI - K_1)^{-1}K_2 + K_4$$  \hspace{1cm} (7.7)

Compared to the TFM of static observer gain (which is a real constant matrix $K$ without complex variable $s$), the TFM (7.7) has the ability to change the frequency characteristics of the correction term $r(t)$ when $r(t)$ is fed back to the forward path (7.3).

\textbf{Remark 7.1.} Before proceeding, it should be pointed out that the dynamic observer contains the classic static observer as a particular case. To see this, selecting the gain matrix as

$$K = \begin{bmatrix} 0 & 0 \\ 0 & K_4 \end{bmatrix}$$  \hspace{1cm} (7.8)

gives $\dot{\xi} = 0$ and

$$v = K_4(y - \hat{y})$$  \hspace{1cm} (7.9)

which is identical to the static observer with $L = K_4$.

By connecting the dynamic feedback (7.4) and the forward part (7.3), the
overall dynamics of the dynamic observer can be rewritten in an augment form:

\[
\begin{cases}
    \begin{pmatrix} \dot{\hat{x}} \\ \dot{z} \end{pmatrix} = 
    \begin{bmatrix} A - K_4 C & K_3 \\ -K_2 C & K_1 \end{bmatrix} 
    \begin{pmatrix} \hat{x} \\ z \end{pmatrix} 
    + \begin{bmatrix} B \\ 0 \end{bmatrix} u 
    + \begin{bmatrix} K_4 \\ K_2 \end{bmatrix} (y - Du) \\
    \hat{y} = [C \\ 0] \begin{pmatrix} \hat{x} \\ z \end{pmatrix} + D u
\end{cases}
\]

(7.10)

Defining \( e(t) = x(t) - \hat{x}(t) \) and subtracting (7.3) from system (7.1) yield

\[
\begin{cases}
    \begin{pmatrix} \dot{e}(t) \\ \dot{z}(t) \end{pmatrix} = 
    \begin{bmatrix} A - K_4 C & -K_3 \\ K_2 C & K_1 \end{bmatrix} 
    \begin{pmatrix} e(t) \\ z(t) \end{pmatrix} 
    + \begin{bmatrix} B_d - K_4 D_d \\ K_2 D_d \end{bmatrix} d(t) 
    + \begin{bmatrix} B_f - K_4 D_f \\ K_2 D_f \end{bmatrix} f(t) \\
    r(t) = [C \\ 0] \begin{pmatrix} e(t) \\ z(t) \end{pmatrix} + D_d d(t) + D_f f(t)
\end{cases}
\]

(7.11)

Equation (7.11) can be rewritten in the following compact form:

\[
\begin{cases}
    \dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{B}_d d(t) + \tilde{B}_f f(t) \\
    r(t) = \tilde{C} \tilde{x}(t) + \tilde{D}_d d(t) + \tilde{D}_f f(t)
\end{cases}
\]

(7.12)

where

\[
\tilde{x} = [e^T(t) \ z^T(t)]^T, \quad \tilde{A} = \begin{bmatrix} A - K_4 C & -K_3 \\ K_2 C & K_1 \end{bmatrix},
\]

\[
\tilde{B}_d = \begin{bmatrix} B_d - K_4 D_d \\ K_2 D_d \end{bmatrix}, \quad \tilde{B}_f = \begin{bmatrix} B_f - K_4 D_f \\ K_2 D_f \end{bmatrix},
\]

\[
\tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \tilde{D}_d = D_d, \quad \tilde{D}_f = D_f
\]

There are various methods to design the dynamic observer parameters \( K_1, K_2, K_3 \) and \( K_4 \), such as eigenstructure assignment [22], and dual controller design approach [128], [108], [122]. In this paper, we study the properties of the dynamic observer zeros first, and propose an approach utilizing the zero assignment technique.
7.3 TFMs of dynamic observer

It can be seen from (7.12), under the condition that the model is adequately accurate, \( r(t) \) is not affected by the system input \( u(t) \), as the process dynamics are canceled in the observer. However, both \( f(t) \) and \( d(t) \) contribute to non-zero \( r(t) \). Assuming the initial conditions are zero, \( s \)-transform of (7.12) gives the TFM relating \( d(t) \) to \( r(t) \)

\[
\tilde{G}_d(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}_d + \tilde{D}_d. \tag{7.13}
\]

Similarly, the TFM relating \( f(t) \) to \( r(t) \) is given by

\[
\tilde{G}_f(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}_f + \tilde{D}_f \tag{7.14}
\]

The whole dynamics of the residual can be expressed as:

\[
\bar{r}(s) = \tilde{G}_f(f(s)) + \tilde{G}_d(d(s)) \tag{7.15}
\]

It can be seen clearly from (7.15) that, due to the existence of \( d(s) \), the residual \( r(t) \) is nonzero even when there is no fault. For a successful fault detection, it is essential to make \( \tilde{G}_d(s) \) small and enlarge \( \tilde{G}_f(s) \).

7.3.1 Poles of dynamic observer

From the simplified expression (7.12), it can be seen that the poles of the observer (7.12) are the roots of the characteristic polynomial \( \det(sI - \tilde{A}) = 0 \). It follows that the complete set of poles coincides with the eigenvalues of the matrix

\[
\tilde{A} = \begin{bmatrix}
A - K_4C & -K_3 \\
K_2C & K_1
\end{bmatrix}. \tag{7.16}
\]

Thus, the stability of the dynamic observer is determined by the matrix \( \tilde{A} \) and can be stated as follows:

**Theorem 7.1.** Given a system (7.1) and the corresponding dynamic observer (7.3)-(7.4), the dynamic observer is stable if and only if all the eigenvalue of \( \tilde{A} \) are in the open left half \( s \)-plane.

**Proof.** The result follows immediately from (7.16). \( \square \)
It is worth to note that, because of the importance of poles, most observer design approaches (e.g., eigenstructure assignment) in literatures concern the positions of poles and ignored the zeros. The free parameters unrelated to poles are selected roughly by optimisation algorithms. Next, we will analyse the zeros of the dynamic observer in order to assign zeros by placing appropriate values to some free parameters.

7.3.2 Zeros of dynamic observer

During the last three decades, considerable research has been done on defining zeros (called transmission zeros or invariant zeros) and deriving their properties. Generally, the transmission zeros are defined in terms of TFM [152]. The transmission zeros are the complex numbers such that the rank of the TFM is locally reduced. It has been shown that, if \( s \) is a zero, then there exists some non zero proportional \( e^{st} \) input vector such that its propagating through the system is blocked [153]. In this paper, this property will be used to attenuate the propagation of \( d(s) \) in (7.15). First of all, the disturbance zeros are defined in an analogous way as the definition of transmission zeros.

**Definition 7.1. (disturbance zeros of the plant)** The disturbance zeros of the plant (7.1) are the transmission zeros of the TFM relating the disturbance \( d(t) \) to the system output \( y(t) \). That is the set of complex numbers \( s \) such that \( G_d(s) \) loses rank locally

\[
Z_1 = \{ s \mid \text{rank } G_d(s) < \min(n_r, n_d) \} \tag{7.17}
\]

where \( n_r, n_d \) are the dimension of the residual and disturbance, respectively, and \( G_d(s) = C(sI - A)^{-1}B_d + D_d \).

Note that these disturbance zeros defined in (7.17) are different from the system (input/output) transmission zeros. The disturbance zeros \( \{ Z_1 \} \) are related with TFM from \( d(t) \) to \( y(t) \), whereas the system transmission zeros are associated with TFM \( C(sI - A)^{-1}B + D \) relating \( u(t) \) to \( y(t) \).

**Definition 7.2. (disturbance zeros of the dynamic observer)**: The disturbance zeros of the observer (7.12) are the transmission zeros of \( \tilde{G}_d(s) \) such that \( \tilde{G}_d(s) \) loses rank locally.

\[
Z_2 = \{ s \mid \text{rank } \tilde{G}_d(s) < \min(n_r, n_d) \} \tag{7.18}
\]
It is worth noting that disturbance zeros $Z_2$ may vary from $Z_1$, as $\tilde{G}_d(s)$ differs from $G_d(s)$. The relationship between these two sets will be given in the following theorem. Before proceeding, an existing lemma shows that zeros are associated with reducing a column or row rank of the (Rosenbrock) system matrix.

**Lemma 7.1.** \cite{(152), (130)} Given a completely controllable and observable system $(A, B, C, D)$, for any complex number $s$, the rank of the (Rosenbrock) system matrix $P(s)$ is equal to the rank of the TFM $F(s) = C(sI - A)^{-1}B + D$ plus $n$.

$$\text{rank } P(s) = \text{rank } F(s) + n$$

where

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$$

Lemma 7.1 can be understood as that $F(s)$ loses rank at a complex frequency $s = z$ if and only if the normal rank of the system matrix $P(s)$ is reduced at $s = z$. A useful conclusion is that the zero set is the same as the set of complex numbers $\{s\}$ at which the system matrix $P(s)$ loses rank locally.

**Theorem 7.2.** If the system $(A, B_d, C, D_d)$ is controllable and observable, the disturbance zeros $Z_2$ of the dynamic observer \cite{(7.12)} are the disturbance zeros $Z_1$ of the plant system \cite{(7.1)} together with the poles of the dynamic observer gain \cite{(7.4)}, namely

$$Z_2 = Z_1 \cup \{\text{eigenvalues of } K_1\} \quad (7.19)$$

**Proof.** According to Lemma 7.1, the disturbance zeros of $G_d(s)$ in system \cite{(7.4)} coincide with those complex values $s$ at which the Rosenbrock system matrix $P_d(s)$ loses rank locally

$$Z_1 = \{s | \text{rank } P_d(s) < n + \min(n_r, n_d)\}, \quad (7.20)$$

where

$$P_d(s) = \begin{bmatrix} sI - A & B_d \\ -C & D_d \end{bmatrix} \quad (7.21)$$

with normal rank $n + \min(n_r, n_d)$.

Similarly, for the dynamic observer \cite{(7.12)}, the disturbance zeros are the set of complex numbers $s$ such that

$$\text{rank } \tilde{P}_d(s) < n + m + \min(n_r, n_d) \quad (7.22)$$
where

\[
\tilde{P}_d(s) = \begin{bmatrix}
    sI - A + K_4C & K_3 & B_d - K_4D_d \\
    -K_2C & sI - K_1 & K_2D_d \\
    -C & 0 & D_d
\end{bmatrix}
\] (7.23)

is the corresponding Rosenbrock system matrix with size \((n+m+n_r) \times (n+m+n_d)\) and normal rank \(n + m + \min(n_r, n_d)\). Hence, the disturbance zeros of \(\tilde{G}_d(s)\) can be expressed as

\[
Z_2 = \{ s | \text{rank} \tilde{P}_d(s) < n + m + \min(n_r, n_d) \} \] (7.24)

The rank of \(\tilde{P}_d(s)\) (7.23) can be calculated by

\[
\text{rank} \tilde{P}_d(s) = \text{rank} \begin{bmatrix}
    sI - A & K_3 & B_d \\
    -K_2C & sI - K_1 & K_2D_d \\
    -C & 0 & D_d
\end{bmatrix} = \text{rank} \begin{bmatrix}
    0 & sI - K_1 & 0 \\
    -C & 0 & D_d
\end{bmatrix} = \text{rank} \begin{bmatrix}
    sI - A & B_d & K_3 \\
    -C & D_d & 0 \\
    0 & 0 & sI - K_1
\end{bmatrix}
\] (7.25)

It follows that the observer disturbance zeros \(Z_2\) are those values of \(s\) for which

\[
\text{rank} \begin{bmatrix}
    sI - A & B_d \\
    -C & D_d
\end{bmatrix} < n + \min(n_r, n_d) \] (7.26)

or/and

\[
\text{rank} [sI - K_1] < m \] (7.27)

Comparing (7.26) and (7.20), one can see that the zeros given by equation (7.26) coincide with \(Z_1\). Then the disturbance zeros of the plant system is a subset of the disturbance zeros of the corresponding dynamic observer.

Equation (7.27) shows that the eigenvalues of \(K_1\) compose another subset of the dynamic observer disturbance zeros. These zeros are the poles of the dynamic gain (7.4).

Thus, the result of the theorem follows.
Theorem 7.2 verifies that the disturbance zeros are invariant in static observer. It means, if $s$ is a disturbance zero of plant (7.1), then $s$ is a zero of the corresponding dynamic/static observer too. Due to the zero invariance, one cannot shift the positions of zeros in the static observer. In dynamic observers, however, the extra disturbance zeros introduced by the dynamic gain can be arbitrarily assigned, even if some of the observer disturbance zeros are invariant from the system disturbance zeros. This is the main implication of Theorem 7.2.

Theorem 7.2 implies that a $m$-th order dynamic system (7.4) introduces $m$ additional transmission zeros in the dynamic observer, and the additional zeros are located at the poles of the dynamic feedback gain. Theorem 7.2 can be understood as a generalization of the well-known SISO dynamic feedback control result that closed-loop zeros are zeros in the forward-path and poles in the feedback-path.

Remark 7.2. From Theorem 7.2 it follows that the PI observer proposed in [123] only introduces disturbance zeros at the origin. This explains why the PI observer achieves a better performance at rejecting step/constant disturbance in the steady state.

Remark 7.3. It is of interest to draw attention to the dynamic observer in [108], where two dynamic systems are adopted to replace two constant gain matrices. It can be seen that, by adding two dynamic feedbacks (one added into $\hat{x}$, one into $\hat{y}$), more parameters and design freedom are provided than that of the present dynamic observer. It is of interest and encouraging to extend the present work by the use of this kind of observers in the future.

Remark 7.4. The proposed dynamic observer is a robust observer, which differs from the Kalman Filter (KF) in two ways: 1) different structures, the KF has the same structure as the static observer, and the dynamic observer extends the static gain matrix to a dynamic system; 2) different performance criteria. The KF aims at minimizing the covariance of state estimation error under stochastic noises. The objective of the robust observer is to enhance robustness to model uncertainties and/or deterministic disturbances. Therefore, the resulting observers show different performances. As reported in literatures [121], the KF works well in rejecting stochastic noises with known covariance, and robust observers achieve a better performance in attenuating deterministic disturbances. Furthermore, the advantages of the dynamic observer are: a) more design freedom; b) by assigning
zeroes, the number of free parameters is reduced by \( m \times m \) and the computation costs are reduced at some extent.

Without claiming neither that the Kalman filter is not useful in fault detection, nor that the (dynamic/static) observer is the best one available, it is felt that, as a different structure, the robust dynamic observer is an alternative, which can give a better robust performance.

### 7.4 Zero-pole assignment procedure

It is well known that poles are insufficient to determine the system behavior that is also greatly affected by zeros. For instance, in a SISO system, if the transfer function has a zero at \( \pm j\omega \), then the magnitude response at frequency \( \omega \) is zero. Thus, the system output of the sinusoidal input \( a \sin(\omega t) \) is zero. According to the theory of system zero \([152], [153]\), if \( s \) is the zero of a MIMO system, then there exists some input (vector) \( u(t) = \beta e^{s t}, (\beta \neq 0) \) such that its propagation through the system is blocked. Since disturbance zeros \( Z_2 \) are defined as the zeros between \( d(t) \) and \( r(t) \), for some specific disturbance, \( d(t) \) can not pass through the observer. One can use this property to obtain disturbance attenuation. Hence, placing one or more disturbance zeros at the origin (or \( j\omega \) on the imaginary axis) will attenuate a step disturbance (or a sinusoidal disturbance \( e^{j\omega t} \), respectively).

Unfortunately, because of the invariance of zeros, the disturbance zeros in static observers can not be changed. Theorem \ref{thm:zero-assignment} however, shows the possibility of zero assignment in dynamic observers by introducing additional zeros. It is also shown that the additional zero is independent from \( K_2, K_3 \) and \( K_4 \). The zero assignment in dynamic observer can be stated as: the additional zeros introduced by \( K_1 \) can be arbitrarily placed to desired positions by assigning the eigenvalues of \( K_1 \) properly. Thus, the limitation of invariant zeros in static observers is overcome and the performance can be improved to some extent.

A further consideration is that the set of zeros must be self-conjugated, such that the resulting matrix \( K_1 \) is real. For a disturbance at frequency \( \omega_i \), the desired zeros should be \( \pm j\omega_i \). Thus, the required number of poles is double of the number of disturbance frequencies. The order of the dynamic feedback \( \ref{eq:dynamic-feedback} \) can be determined by

\[
m = 2n_{\omega_d}
\]  \( \text{(7.28)} \)
where \( n_{\omega_d} \) is the number of disturbance frequencies. Hence, the sizes of the parameter matrices \( K_1, K_2, K_3, K_4 \) are determined accordingly. For pole assignment, the matrix \( \tilde{A} \) can be decomposed as

\[
\tilde{A} = \begin{bmatrix}
A - K_4 C & -K_3 \\
K_2 C & K_1
\end{bmatrix}
= \begin{bmatrix}
A & 0 \\
0 & K_1
\end{bmatrix} + \begin{bmatrix}
-K_4 & -K_3 \\
K_2 & 0
\end{bmatrix} \begin{bmatrix}
C & 0 \\
0 & I
\end{bmatrix}
\] (7.29)

where \( A, C \) is known and \( K_1 \) is determined by the zero assignment. The task of pole assignment is to find an appropriate parameter matrix

\[
\begin{bmatrix}
-K_4 & -K_3 \\
K_2 & 0
\end{bmatrix}
\]

so that the eigenvalues of \( \tilde{A} \) are assigned to some areas on the left-half \( s \)-plane.

**Remark 7.5.** Note that, as the magnitude response in the neighborhood of the pole shoots up to infinity, the pole regions should be far from the disturbance frequency \( \pm j\omega \) to avoid their effects of enlarging the magnitude response at \( \pm j\omega \). Similarly, because the major frequency components of abrupt/incipient faults are at low frequency [17], the poles of the fault TFM \( \tilde{G}_f(s) \) should be close to the origin, such that the magnitude response at low frequency is enlarged and the effects of \( f(t) \) in residual \( r(t) \) can be enhanced.

**Remark 7.6.** The benefit of zero assignment is that a better trade-off is achieved between the design freedom and computation complexity. This would be twofold: 1) making use of the design freedom provided by dynamic observers to improve the disturbance attenuation performance; 2) reducing the computation complexity of dynamic observer design. Particularly, it is true for the issue of ‘curse of dimensions’ in parameter optimisation. Compared to static observers where only one \( n \times r \) matrix need to be optimised, the number of free parameters in dynamic observers are \((m + n) \times (m + r)\). For most recent dynamic observer design approaches, these free parameters are selected roughly by optimisation algorithms [123], [128]. The possibility of being trapped in local minima increases as the dimension of parameter increases. With the aid of zero assignment by specifying the matrix \( K_1 \), the dimension of the search space in parameter optimisation is reduced by \( m \times m \).

In summary, the zero assignment technique in dynamic observer design, on one side, is able to attenuate the disturbance further by assigning zeros close
to the disturbance frequency, and, on the other side, to reduce the computation complexity by diminishing the search space.

The zero assignment solution to the dynamic robust fault detection observer design can now be stated as follows:

*Given a system (7.1) corrupted by \(d(t)\), if the main frequency contents of residuals \(r(t)\) can be estimated at \(w_i, (i = 1, 2, \ldots, n_d)\), then, (a) assign the zeros of dynamic observer to \(\pm jw_i, (i = 1, 2, \ldots, n_d)\); (b) place the eigenvalues of \(\tilde{A}\) in the left half s-plane and (c) minimize the following performance index

\[
J = \sum_{i=1}^{n_d} W_i \left| \tilde{G}_d(s) \right|_{2,s=jw_i} \rho + \left| \tilde{G}_f(s) \right|_{\infty}
\]  

(7.30)

to give the optimal gain matrix \(K_2, K_3\) and \(K_4\). Therefore, residual \(r(t)\) is an optimal detection of \(f(t)\).*

Here, \(W_i\) is the weighting factor selected according to the distribution of the disturbance, \(\| \cdot \|_{2,s=jw_i}\) denotes the 2-norm of a TFM at \(s = jw_i\), \(\| \cdot \|_{\infty}\) denotes the \(H_\infty\) norm of a TFM and \(\rho\) is a small real number to guarantee the denominator will not be zero. The minimization of numerator \(\sum_{i=1}^{n_d} W_i ||\tilde{G}_d||\) is for attenuating disturbances (in another words, to enhance the robustness to disturbances); the minimization of \(1/(\rho + ||\tilde{G}_f||_{\infty})\) (equivalently, maximization of \(||\tilde{G}_f||_{\infty})\) is to enhance the effects of faults in residuals \(r(t)\).

### 7.5 Application and results

To illustrate the proposed dynamic fault detection observer design approach, this section considers robust fault detection of a gas turbine engine. The model of the considered engine is:

\[
\begin{cases}
\dot{x}(t) = \begin{bmatrix} 
-0.943 & 0.1601 \\
3.9439 & -3.234 
\end{bmatrix} x(t) + \begin{bmatrix} 
86.794 & 40.312 \\
154.691 & 81.275 
\end{bmatrix} u(t) \\
y = \begin{bmatrix} 
1 & 0 \\
0 & 1 
\end{bmatrix} x(t)
\end{cases}
\]  

(7.31)
The disturbance model is assumed as

\[ B_d = B_f = B \quad D_d = D_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \] (7.32)

where the fault matrix \( B_f = B \) for actuator faults. Note that, \( B_d = B \) too, as it is common in the industrial applications where disturbances enter the system by corrupting the input signal. A further advantage is that the estimation of disturbance matrix \( B_d \) is avoided. It can be verified that the system (7.31) and \((A, B_d, C, D_d)\) is controllable and observable. Then the condition of Theorem 7.2 is met.

It is worth noting that, in this configuration \((B_d = B_f = B)\), the widely used \( H_\infty / H_\infty \) static observer design may fail without weighting functions, as the TFMs \( \tilde{G}_f(s) \), \( \tilde{G}_d(s) \) are the same and the performance index \( \| \tilde{G}_d(s) \|_\infty / \| \tilde{G}_f(s) \|_\infty \) is always equal to 1. This can be seen clearly from Figure 7.2.

![Figure 7.2: The largest singular values and smallest singular values of \( \tilde{G}_d(s) \) and \( \tilde{G}_f(s) \), respectively.](image)

The disturbances injected to the systems are cyclo-stationary signals:

\[ d(t) = \begin{pmatrix} 0.25[\sin(5t) + \sin(10t + \pi / 4)] + n_1(t) \\ 0.25[\cos(5t) + \sin(10t + \pi / 3)] + n_2(t) \end{pmatrix} \] (7.33)

where \( n_1(t) \) and \( n_2(t) \) are mutually independent white noises \( N(0, 0.01) \) with zero mean and variance 0.01. Figure 7.3 shows the disturbances. In the simulation, the inputs \( u(t) \) are unit step signals and both the amplitudes of \( d_1(t) \) and \( d_2(t) \) are over 50% of the amplitudes of input signals. Moreover, the disturbances contain mainly low frequency components. Attenuating low frequency disturbances is
given higher priority, because common step/incipient faults are mainly at low frequencies and most of industrial systems behave as low-pass observers. Hence, it is more challenging to attenuate low frequency disturbances.

![Figure 7.3: Quasi-stationary disturbances](image)

Following the procedure in the preceding section, now a dynamic observer is designed for this system.

**Step 1.** In order to estimate the disturbance frequency, a static observer is first constructed via $K_0 = \text{place}(A^T, C^T, [-1, -1])^T$. A 1024-point FFT is employed to calculate the spectrum of its residual $r(t)$. Figure 7.5 provides an illustration of the spectrum of the actual disturbance and its estimate via the residual. From the plot on the right of Figure 7.5, it can be seen that $r(t)$ has two main components corresponding to the disturbance frequencies $\omega = \{5.0, 10\}$ (rad/sec). This
estimated frequencies agree with the actual disturbance frequencies. Hence, set $n_d = 2$, $\omega_i = \{5.0, 10\}$ and the weighting factors $W_i = 1, i = 1, 2$

**Step 2.** It follows that the desired zero positions are $\pm 5j$, $\pm 10j$ and the order of the dynamic gain is $m = 4$. Then the dynamic observer structure is $K_1 \in \mathbb{R}^{4 \times 4}, K_2 \in \mathbb{R}^{4 \times 2}, K_3 \in \mathbb{R}^{2 \times 4}, K_4 \in \mathbb{R}^{2 \times 2}$. Let

$$K_1 = \begin{bmatrix} 0 & -5 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 10 & 0 \end{bmatrix} \quad (7.34)$$

to assign zeros to $\{\pm 5j, \pm 10j\}$.

**Step 3.** Set the desired regions of poles according to: (a) their real parts are less than $-1$, and (b) their imaginary parts are close to real axis as much as possible. The reason for (b) is that, if the pole is on real axis without imaginary part, its effects on imaginary frequencies may be reduced. An equivalent statement was presented in Remark 7.

**Step 4.** Set the initial values of $K_2, K_3, K_4$ randomly and use the algorithm *fmincon* (Optimization Toolbox in Matlab). The optimal solution is

$$K_2 = \begin{bmatrix} 0.0727 & 3.7092 \\ 2.5856 & -5.3730 \\ -0.2838 & 5.2500 \\ -6.2984 & 0.0765 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 18.5227 & -5.0874 \\ 5.9855 & 16.4604 \end{bmatrix} \quad (7.35)$$


which gives the resulting dynamic observer

$$\bar{A} = \begin{bmatrix} -19.47 & 5.248 & -1.72 & 4.523 & 27.18 & -6.04 \\ -2.042 & -19.7 & 3.565 & 12.41 & 7.284 & 8.843 \\ 0.072 & 3.709 & 0 & -5 & 0 & 0 \\ 2.586 & -5.373 & 5 & 0 & 0 & 0 \\ -0.284 & 5.250 & 0 & 0 & 0 & -10 \\ -6.298 & 0.077 & 0 & 0 & 10 & 0 \end{bmatrix} \quad (7.36)$$
\[
\tilde{B}_d = \tilde{B}_f = \begin{bmatrix}
86.794 & 40.312 \\
154.691 & 81.275 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\] (7.37)

\[
\tilde{C} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \tilde{D}_d = 0, \quad \tilde{D}_f = 0
\] (7.38)

And the six poles of the dynamic observer are:

\[
\{-16.8934, -15.8096, -3.4574 - 1.0 \pm 0.2617j, -1.0\} 
\] (7.39)

As discussed before, since the disturbance(fault) distribution matrices \(B_d = B_f = B\), the resulting TFMs \(\tilde{G}_f(s)\) and \(\tilde{G}_d(s)\) are identical, as shown in equation (7.40).

\[
\tilde{G}_f(s) = \tilde{G}_d(s) = \frac{\begin{bmatrix}
H_{11}(s) & H_{12}(s) \\
H_{21}(s) & H_{22}(s)
\end{bmatrix}}{(s^2 + 1.833s + 0.8447)(s^2 + 2.344s + 1.382)}
\] (7.40)

where

\[
H_{11}(s) = 86.79(s + 14.75)(s^2 + 3s + 18.2)(s^2 + 11.3s + 75.3) \\
H_{12}(s) = 154.7(s + 16.63)(s^2 + 2.41s + 6.62)(s^2 - 0.713s + 35.74) \\
H_{21}(s) = 40.31(s + 14.4)(s^2 + 2.27s + 17.65)(s^2 + 13.6s + 87.26) \\
H_{22}(s) = 81.28(s + 16.65)(s^2 + 2.225s + 6.615)(s^2 - 0.42s + 34.78)
\]

It is easy to verify that the normal rank of TFM (7.40) is 2 and it reduces to 1 when \(s = \{\pm 5j, \pm 10j\}\). Therefore, the disturbance zeros of TFM (7.40) are \(\{\pm 5j, \pm 10j\}\) which is consistent with Theorem 7.2.

For fair comparison to the conventional static observer, a \(H_{\infty}/H_{\infty}\) static observer \(L_1\) is designed by taking use of weighting functions. Since the zeros are of interest, the effects of different poles should be reduced as much as possible.
Hence, the pole regions are restricted to the similar area as the dynamic observer. The resulting $H_\infty/H_\infty$ observer gain matrix is

$$L_1 = \begin{pmatrix} 4.0827 & -3.8591 \\ 7.9752 & -6.2600 \end{pmatrix} \quad (7.41)$$

which gives the observer poles at $-1.0 \pm 0.0082j$.

A static observer is also designed by using $\text{place()}$ function provided by Matlab which gives the static gain matrix

$$L_2 = \begin{pmatrix} 0.0574 & -0.1016 \\ 4.2056 & -2.2347 \end{pmatrix} \quad (7.42)$$

with poles at $-1.0 \pm 0.2617j$. Note that, because the static observers do not change the system order, they have only two poles.

The magnitude responses of $\tilde{G}_f(s)$ and $\tilde{G}_d(s)$ of these three observers (dynamic observer (solid), static observer $L_1$ (dotted) and $L_2$ (dash-dot)) are depicted in Figure 7.5, where $\tilde{G}_d(s)$ and $\tilde{G}_f(s)$ are identical due to $B_d = B_f, D_d = D_f$.

![Figure 7.5: Comparison of magnitude responses of TFMs $\tilde{G}_f(s)$ and $\tilde{G}_d(s)$](image)
It can be seen that, compared to those of static observers, the dynamic observer has dips around the disturbance frequencies in the magnitude response. These dips contribute the improvement on attenuating the band-limited disturbances. Furthermore, in the dynamic observer, the magnitude around zero frequency (where the main components of the fault are) is greater than that of static observers, which enhances the effects of the fault in residuals.

A. Residuals without fault

In this simulation, no fault is introduced. Figure 7.6 shows the norms of $r(t)_{\text{dyn}}$, $r(t)_{L_1}$ and $r(t)_{L_2}$, which are the residuals of our dynamic observer, the $H_\infty/H_\infty$ observer $L_1$ and static observer $L_2$, respectively. It can be seen that $\|r(t)\|_{\text{dyn}}$ has a large overshoot at the beginning due to the transient process of the dynamic observer. In the steady state, the disturbance attenuation in dynamic observer is more significant than that of $L_1$, $L_2$. In the steady state, the maximum magnitude of $\|r(t)\|_{\text{dyn}}$ is below 8, however that of $\|r(t)\|_{L_1}$, $\|r(t)\|_{L_2}$ is nearly 20.

![Figure 7.6](image_url)

Figure 7.6: Fault free residuals corresponding to the dynamic observer, the $H_\infty/H_\infty$ observer $L_1$ and static observer $L_2$, respectively.

The disturbance attenuation performance is also evaluated in terms of the root mean square error (RMSE) and the standard deviation (STD):

$$RMSE = E\{\|y(t) - \hat{y}(t)\|\} = E\{\|r(t)\|\}$$

$$STD = \sqrt{E\{(\|r(t)\| - RMSE)^2\}} \quad \text{(7.43)}$$
where $\|y\|$ denotes a vector norm $\sqrt{\sum_i |y_i|^2}$ and $E\{}$ denotes the mathematical expectation. The RMSE and STD values are also shown in Figure 7.6.

B. Residuals of abrupt actuator faults

In order to simulate two successive abrupt actuator faults at two input channels respectively, the fault function is represented as $f_a(t) = [f_1(t) \ f_2(t)]^T$ and

$$f_1(t) = \begin{cases} 0 & (t < 20) \\ 0.05 & (t \geq 20) \end{cases} \quad f_2(t) = \begin{cases} 0 & (t < 30) \\ 0.05 & (t \geq 30) \end{cases}$$

![Residuals of abrupt actuator faults](image)

Figure 7.7: Residuals of the dynamic observer, $H_\infty/H_\infty$ observer $L_1$ and static observer $L_2$, respectively, for detecting abrupt actuator faults

The residuals of the dynamic observer, static observer $L_1$ and $L_2$ are depicted in Figure 7.7, where two step increases in $\|r(t)\|_{dyn}$ can be seen clearly within 1 second after each fault occurs. The observers $L_1$ and $L_2$, however, fail to detect such abrupt faults. Although the peak values of the residuals of these static observers present step increases, there is no clear interval between the normal residuals and faulty residuals.
C. Residuals of an incipient actuator fault

In this trial, the gradual actuator fault injected to the input signal is \( f_a(t) = [f_1(t) f_2(t)]^T \), where \( f_1(t) \) occurs at 20 second with slope rate 0.0025

\[
f_1(t) = \begin{cases} 
0 & (t < 20) \\
0.0025(t - 20) & (k \geq 20)
\end{cases}; \quad f_2(t) = 0 \tag{7.44}
\]

The corresponding residuals are shown in Figure 7.8. To illustrate the fault detection performance, the plant outputs under such a fault are also plotted in Figure 7.8. Due to the large output values and the small fault size, the changes in the outputs is difficult to distinguish. However, this fault can be detected clearly from the dynamic observer residuals. \( \|r(t)\|_{dyn} \) responses the incipient fault (7.44) with a fairly straight line increasing at about 4 units per second. Whereas \( \|r(t)\|_{L_1} \) is with significant disturbances and its increase rate is only 0.7 unit per second. This comparison verifies that the dynamic observer is able to detect smaller gradual fault earlier and avoid missed alarm as much as possible.

![System output and residuals under incipient actuator fault](image)

Figure 7.8: Plant outputs and residuals under an incipient actuator fault at the first input channel.

7.6 Summary

In this chapter, a systematic study of the zeros of dynamic observer is presented. The possibility of zeros assignment in dynamic observer design has been proved and the capacities for fault detection are illustrated by simulation.

The proposed dynamic observer differs from the classical static observer whose gain matrix is a constant coefficient matrix. A dynamic gain is introduced into the dynamic observer and it is proved that a dynamic observer introduces additional zeros and shifts the system poles from \( eig(A) \) to \( eig(\tilde{A}) \). This technique
brings more freedom for observer design and these additional degrees of design freedom are used to assign the additional zeros to desired places for attenuating disturbances further. The dynamic design method proposed here takes use of the zero-assignment freedom provided by the dynamic gain and combine the pole assignment method for a better disturbance attenuation performance.

In the application results, even the low frequency disturbances (5 rad/sec) can be attenuated by the propose observer and both the actuator and sensor faults are detected. Simulation results have shown that the proposed zeros assignment method comes with reasonably good disturbance attenuation. Hence, we are able, on one side, to formulate and get a better observer in the sense of robustness and disturbance rejection, and, on the other side, to obtain new insight into the observer construction for fault detection.
Bibliography


Appendix A

Models

This appendix contains the model and relevant information for the systems used in this PhD study. This appendix aims to provide a background of the dynamic modelling and condition monitoring of aero gas turbine engines.

A.1 Gas Turbine Engine

Gas Turbine Engine (GTE) technology was simultaneously proposed in Britain and in Germany in the early 1930s. In the early days, heuristic techniques are widely used in the control systems of gas turbines. In 1970s and 1980s, with the advent of digital signal processing technology, hydraulic systems were replaced with digital systems [45]. Dramatically increased computing power enables advanced control techniques and a greater level of health monitoring to be used. Indeed, health monitoring and aftermarket maintenance are seen as a key commercial driver in aero engine business for the future. For instance, payment for buying an engine now depends on the usage of the engine on a per engine flight hour (EFH) and per engine flight cycle (EFC).

This section is just an introduction to fundamentals of gas turbine engines and discussion of the modelling of gas turbine engines. It presents an overview of GTE modelling and condition monitoring. Emphasis is given on techniques for dynamic modelling of the Spey engine.
A.1.1 Background of the Spey

The Spey is a low-bypass turbofan engine from Rolls-Royce that has been in widespread service for over 30 years. As illustrated in Figure A.1, One variant of the Rolls-Royce Spey is with a four-stage low pressure (LP) compressor and 12-stage high pressure (HP) compressor driven by a two-stage LP turbine and two-stage HP turbine respectively. These rotate at different speeds, which are denoted by $N_{lp}$ and $N_{hp}$, respectively. The core element is the gas generator which is typically made up of five major components: an air inlet duct, a compressor, a combustion chamber (or combustor), a turbine and an exhaust nozzle. In addition to the five major components, the engine is equipped with accessory parts: a fuel system, a starting system, a cooling system, a lubrication system and an ignition system.

The air first enters the intake section, then is compressed by a series of rotating and stationary airfoils moving the air. The resultant energy transfer leads to a rise in pressure, temperature and density of the air. The pressure increases as much as 20 times or more than the pressure at the front air intake.

On discharge from the LP compressors, only part of the air passes through the HP compressor to the combustion chamber, while the remainder by-passes the engine core. The ratio between the mass flow of air in the by-pass duct and in the core is termed the by-pass ratio. The air enters the combuster, and mixes with the sprayed fuel. The air-fuel mixture is then burns at a relatively constant pressure.
Appendix A.  

thus adding a large amount of energy to the airflow. The combustion process leads to a steep increase in temperature, whereas pressure remains virtually constant.

The energy is absorbed from the hot gases in the turbines, where the internal energy of gas stream is converted into mechanical work to drive the HP compressor and LP compressor.

After leaving the turbines, the hot gases mix with the 'cold' gases from bypass duct. Because there is still enough pressure remaining in the mixed gases, the gases are forced through the exhaust duct and jet nozzle at the rear of the engine at a very high speed.

The engine’s thrust comes from taking a large mass of air in at the front end and expelling it from the rear nozzle at a much higher speed than it had when it entered the compressor.

The GTE nomenclature is briefly summarised in Table A.1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{PLA}$</td>
<td>pilot’s lever angle (PLA)</td>
</tr>
<tr>
<td>$A_9$</td>
<td>nozzle area</td>
</tr>
<tr>
<td>$W_f$</td>
<td>fuel mass flow rate</td>
</tr>
<tr>
<td>$H$</td>
<td>altitude</td>
</tr>
<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$N_l$</td>
<td>low pressure shaft speed</td>
</tr>
<tr>
<td>$N_h$</td>
<td>high pressure shaft speed</td>
</tr>
<tr>
<td>$p_0$</td>
<td>static pressure at the nozzle outlet</td>
</tr>
<tr>
<td>$p_0$</td>
<td>ambient pressure</td>
</tr>
<tr>
<td>$m_i$</td>
<td>mass of the airflow at $i$-th section</td>
</tr>
<tr>
<td>$\dot{m}_i$</td>
<td>mass flow rate at $i$-th section</td>
</tr>
<tr>
<td>$p_i^*$</td>
<td>total pressure at $i$-th section</td>
</tr>
<tr>
<td>$t_i^*$</td>
<td>total temperature at $i$-th section</td>
</tr>
</tbody>
</table>

A.1.2 Modelling of Gas Turbine Engine

Formally, the characteristics of a GTE can be described by a group of mathematical equations. Engine models are required both at the design stage and during engine’s operational life. Different models of GTE are set up for different tasks, e.g., the detailed nonlinear thermodynamic model (performance-based model) for engine design, nonlinear static/dynamic model for demonstration/tuning/production, linear dynamic models for design and analysis of control systems,
real-time piecewise linear model for condition monitoring. Generally, from the view point of control design, a model falls into: (1) Nonlinear static model (2) Nonlinear dynamic model (3) Linear dynamic model.

**Nonlinear static models**

The whole engine models (thermodynamic models) are set up by building the performance functions for describing operation of each component. The performance function of \( i \)-th component can be expressed as

\[
\theta_i = F_i (\theta_{i-1})
\]

where \( F_i \) denote the dynamic process taking place at \( i \)-th stage, \( \theta_i = [p^*_i, t^*_i, \dot{m}_i] \) is the performance parameters at \( i \)-th stage.

Furthermore, a group of equations can be formed by connecting these equations on the conservation laws: (1) the conservation of matter; (2) the conservation of energy. The physical laws are expressed in the form of air/gas flow balance, energy balance in the combustion chamber and jet nozzle, power balance between the compressor and turbine, etc.

The close loop GTE system are complemented by the control program equation specifying the control factors \( W_f \) (fuel flow) and \( A_9 \) (nozzle area).

\[
W_f = f_1 (\alpha_{PLA}, t_0, p_0, t^*_9, N_h, N_l, ...)
\]
\[
A_9 = f_2 (\alpha_{PLA}, t_0, p_0, t^*_9, N_h, N_l, ...)
\]

Based on the analysis of physical principles, four types of variables are involved in the GTE model:

(1) the state variables \( X \): e.g., \( X = [W_a, W_g, T^*_c, T^*_g, W_n] \);
(2) the control input variables \( U \): e.g., \( U = [W_f, A_9] \);
(3) the output/performance variables \( Y \): e.g., \( Y = [N_l, N_h, ...] \);
(4) the environment/flight condition variables \( V \): e.g., \( V = [M, H, p_0, t_0] \).

Then the nonlinear model of gas turbine becomes

\[
f_X (X, U, V) = 0
\]
\[
Y = f_Y (X, U, V)
\]
This is the so-called *Detailed Nonlinear Static Model*. This model is applicable only when the engine works at the steady state.

**Nonlinear dynamic model**

In order to describe the transition process (e.g., acceleration, deceleration), the static equations (A.4) need to be differentiated.

\[
\dot{X} = F_X(X, U, V) \\
Y = F_Y(X, U, V)
\]  (A.4)

These differential equations describe the accumulation of the gas energy and mass within the major components, and the changes of shaft speeds. This model is the so-called *Detailed Nonlinear Dynamic Model* and allows study of dynamic properties of a turbine engine.

**Linear dynamic models**

The detailed nonlinear static/dynamic model is the best mathematical model to represent the GTE. But it involves nonlinear equations and it is difficult to design the controller and analyse the stability from the nonlinear models. Two basic methods exist to obtain the linear dynamic model [45].

1. **Physical Principles Modelling (linearisation modelling)** This is an intuitive method to get a linear model by linearising the nonlinear model.

2. **System Identification (or Experimental Modelling)** This is to identify the model parameters from recorded inputs and outputs with the aid of a priori known model structure.

Since nonlinear models of GTEs are differentiable continuous functions during its normal operation [45], it is possible to approximate the nonlinear dynamics by a group of linear dynamic models with sufficient accuracy in a neighbourhood of a set of corresponding steady-state operating points.

**A.1.3 Modelling of the Spey**

As discussed above, a linear dynamic model can be obtained from linearising the nonlinear dynamic model. In the case of the Spey, the resulting linear model
is with five inputs, ten outputs and fifteen states [56]. It was shown that the fifteen states linear model can be reduced to a linear model with the same order as the number of engine shafts [52]. This is because some states of the higher order model are linked with rates of change of pressure within the compressor and turbine volumes. And these states have very fast time constants, because the thermodynamic processes in the gas stream have time constant of 20 ms or less, and the combustion time constant was found to be in the order of 15 ms [52]. While the shaft time constants are in the order of 0.3-4.4s. Thus the higher order model can be reduced to a two state model with the states of shaft speeds.

Furthermore, the SISO transfer function models relating \( N_{lp} \) or \( N_{hp} \) to fuel flow \( W_f \) can be obtained from each of the shafts by evaluating the transfer function matrix.

The engine can be controlled by varying the rate of fuel flow, the angle of the inlet guide vanes, the reheat nozzle area, the reheat fuel flow and the low pressure compressor bleed valve position. However, when the engine works around some operating point, the nozzle area and inlet guide vanes are fixed and the engine is controlled by the fuel flow. In this context, the reheat system is assumed inoperative, the compressor bleed valve closed. Hence, for on-board condition monitoring, the shaft speeds are regarded as the primary outputs, from which the internal engine pressures and thrust can be calculated. Because the engine performance is linked with the shaft speeds closely, attention is paid to modelling the dynamic relationship between these shaft speeds ([\( N_{lp}, N_{hp} \)]) and the fuel flow \( W_f \).

Using the guidelines above, a second order state space model relating shaft speeds \([N_{hp}, H_{lp}]\) to the input fuel flow \( W_f \) is determined in the discrete time domain.

\[
\begin{align*}
\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} &= \begin{bmatrix} 0.9769 & 0.0038 \\ 0.0936 & 0.9225 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} 2.1521 \\ 8.8186 \end{bmatrix} W_f(k) \\
\begin{pmatrix} N_{hp}(k) \\ N_{lp}(k) \end{pmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}
\end{align*}
\]

(A.5)

with a sampling frequency of 40 Hz. This model was obtained at 75% maximum speed. Unless stated otherwise, the above state space model will be used throughout this work.